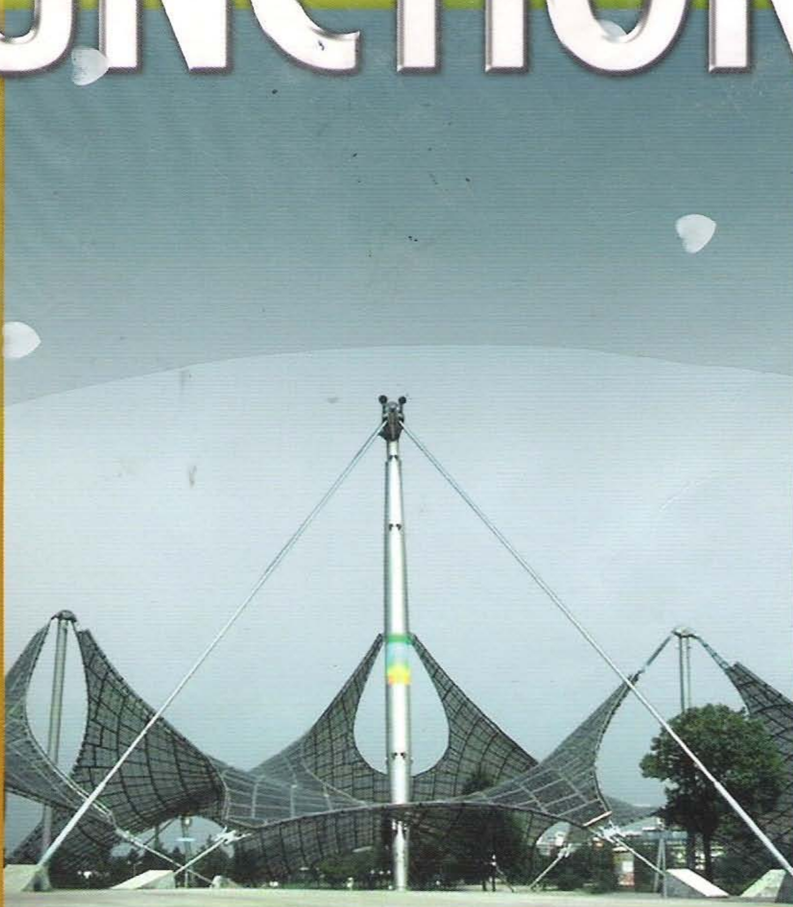
FUNCTIONS



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FUNCTIONS

Cem Giray

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PREFACE

To the Teacher

This is an introductory book covering functions and graphs together with some applications. Functions are used in ordinary life applications, including economics and engineering. Although we do not face functions directly, problems that we face are most frequently solved by modeling them as a function.

This book is divided into four sections. The first section, relations, prepares the student for the concept of functions and introduces the cartesian coordinate system which will be consistent throughout the remaining part of the book. The heart of the book is the second section in which we study the main concepts of functions. The third section goes more in depth with further operations of functions in a more arithmetic and theoretical way. The last section concentrates on visualization of functional terminology throughout the graphs and introduces a very important technique called transformation which will be necessary in future concepts of algebra and mathematical analysis. Teachers may choose four different plans of teaching (sections 1+2, sections 1+2+3, sections 1+2+4, or sections 1+2+3+4) enabling you to use the book depending on the level of the students and the total hours allowed.

The language of this book is more student-friendly rather than purely mathematical. Deep mathematical notation is especially avoided in order to not lead to confusion. The book tries to explain the topic as a teacher would explain it in the classroom, giving examples and exercises which prompt the student to think for him or herself. Most of the examples do not require complex calculations and a large amount of the exercises are about plotting graphs which target analytical skills. With the help of effective use of colors, illustrations and pictures visualization of the material is effectively improved to aid in students' understanding the book, especially the examples and the graphs.

The book follows a linear approach, with material in the latter sections building on concepts and math covered previously in the text. For this reason, there are several self-test 'Check Yourself' sections that check students' understanding of the material at key points. 'Check Yourself' sections include a rapid answer key that allows students to measure their own performance and understanding. Successful completion of each self-test section allows students to advance to the next topic.

Each section is followed by a number of exercises. More difficult problems are denoted by a single or double star, where a single star means problems for upper-intermediate level students, and the double star problems are for an advanced level. Following every section we discuss an activity or project related to the material covered. The topics are Tessellations, Cryptography and the game of Battleship. These sections can be used as term projects to increase the students' understanding of the topic.

The book ends with review materials, beginning with a brief summary of the chapter highlights. Following these highlights is a concept check test that asks the student to summarize the main ideas covered in the book. Following the concept check, review tests cover material from the entire book.

Acknowledgements

Many friends and colleagues were of great help in writing this textbook. A number of people need to be recognized and thanked for their contributions, including Mustafa Kırıkçı at Zambak Publications, Serdar Çam and Şamil Keskinoğlu for their typesetting and design.

Finally, I would like to thank my wife for her support and patience during my work on this book.

Cem Giray

To the Student

This book is designed so that you can use it effectively. Each section has its own special color that you can see at the bottom of the page.

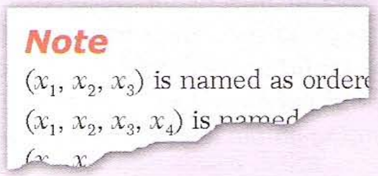
Different pieces of information in this book are useful in different ways. Look at the types of information, and how they appear in the book:

Section 1 *Relations*

Section 2 *Introduction to Functions*

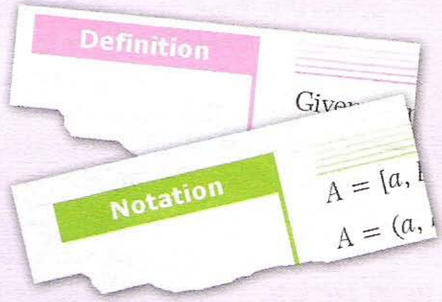
Section 3 *Operations on Functions*

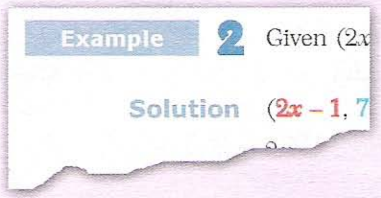
Section 4 *Sketching Graphs of Functions*



***Notes** help you focus on important details. When you see a note, read it twice! Make sure you understand it.*

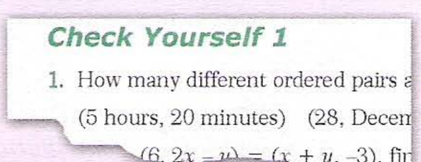
***Definition** boxes give a formal description of a new concept. Notation boxes explain the mathematical way of expressing concepts. The information in these boxes is very important for further understanding and for solving examples.*



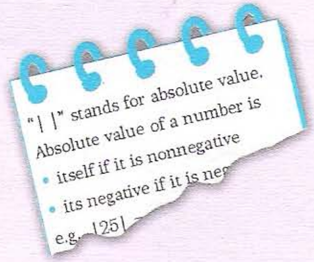


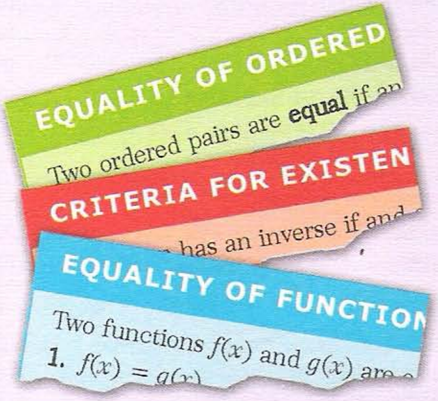
***Examples** include problems related to the topic and their solution, with explanations. The examples are numbered, so you can find them easily in the book.*

***Check Yourself** sections help you check your understanding of what you have just studied. Solve them alone and then check your answers against the answer key provided. If your answers are correct, you can move on to the next section.*



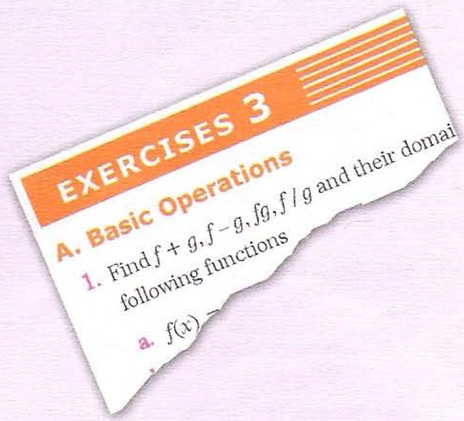
A small notebook in the left or right margin of a page reminds you of material that is related to the topic you are studying. Notebook text helps you to remember the math you need to understand the material. It might help you to see your mistakes, too! Notebooks are the same color as the section you are studying.

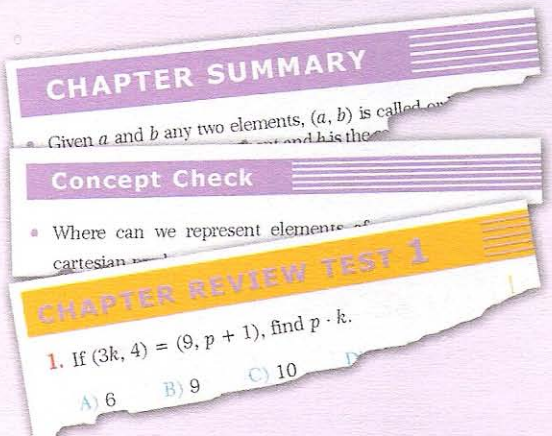




Special windows highlight important new information. Windows may contain formulas, properties, or solution procedures, etc. They are the same color as the color of the section.

***Exercises** at the end of each section cover the material in the whole section. You should be able to solve all the problems without any special symbol. (✪) next to a question means the question is a bit more difficult. (✪✪) next to a question means the question is for students who are looking for a challenge! The answers to the exercises are at the back of the book.*





***The Chapter Summary** summarizes all the important material that has been covered in the chapter. The **Concept Check** section contains oral questions. In order to answer them you don't need a paper or pen. If you answer the Concept Check questions correctly, it means you know that topic! The answers to these questions are in the material you studied. Go back over the material if you are not sure about an answer to a Concept Check question. Finally, **Chapter Review Tests** are in increasing order of difficulty and contain multiple choice questions. The answer key for these tests is at the back of the book*

CONTENTS

INTRODUCTION

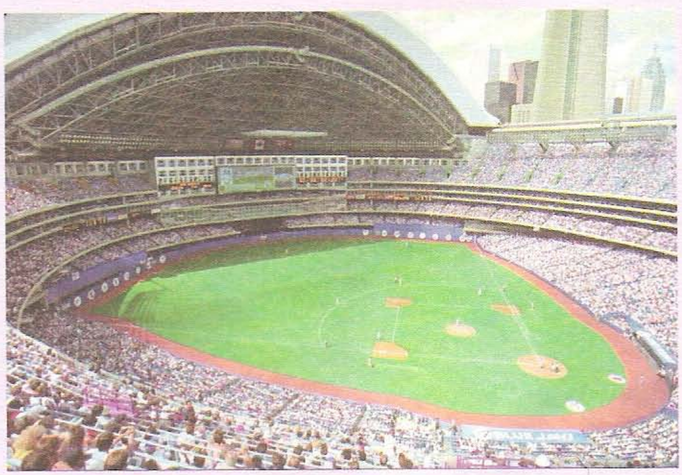
Functions are one of the most important ideas in mathematics, and plays a central role in many areas of mathematics. However, the history of functions is not explained in today's math textbooks; perhaps because of the disagreements of the development of functions and where it came from.

an ancient
Babylonian value table

Many people believe that the history of functions began with Descartes and his use of the coordinate system. Others believe it began at 2000 B.C. with the Babylonians and their use of various value tables such as reciprocals, squares, square roots, cubes, and cubic roots. Other opinions state that functions began with ancient astronomical calculations.

In ancient times, problems between two things were studied and solved, but no generalized ideas of these problems were formed. Therefore, quantities were described verbally or graphically instead of using a formula. The problems they were solving were in fact functions, but they lacked the word "function" thus making it impossible to express these ideas as formulas or analytical expresssions.

In the fourteenth century functions were expressed geometrically. During this time more functions and natural phenomena like heat, light, density, distance, velocity and acceleration were studied. The ideas that developed centered around ideas of independent and dependent variable quantities, but definitions of these were not given. A function was defined by a verbal description of its specific property or by a graph. All quantities and their relations were represented by geometrical forms like straight lines and segments. But the word "function" was still not used.



Architectural structures consist of modelled form of functions.

Leonhard Euler

Gottfried Wilhelm von Leibniz

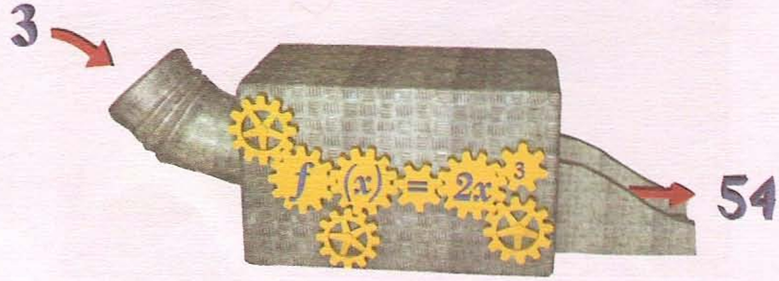
René Descartes

Lejeune Dirichlet

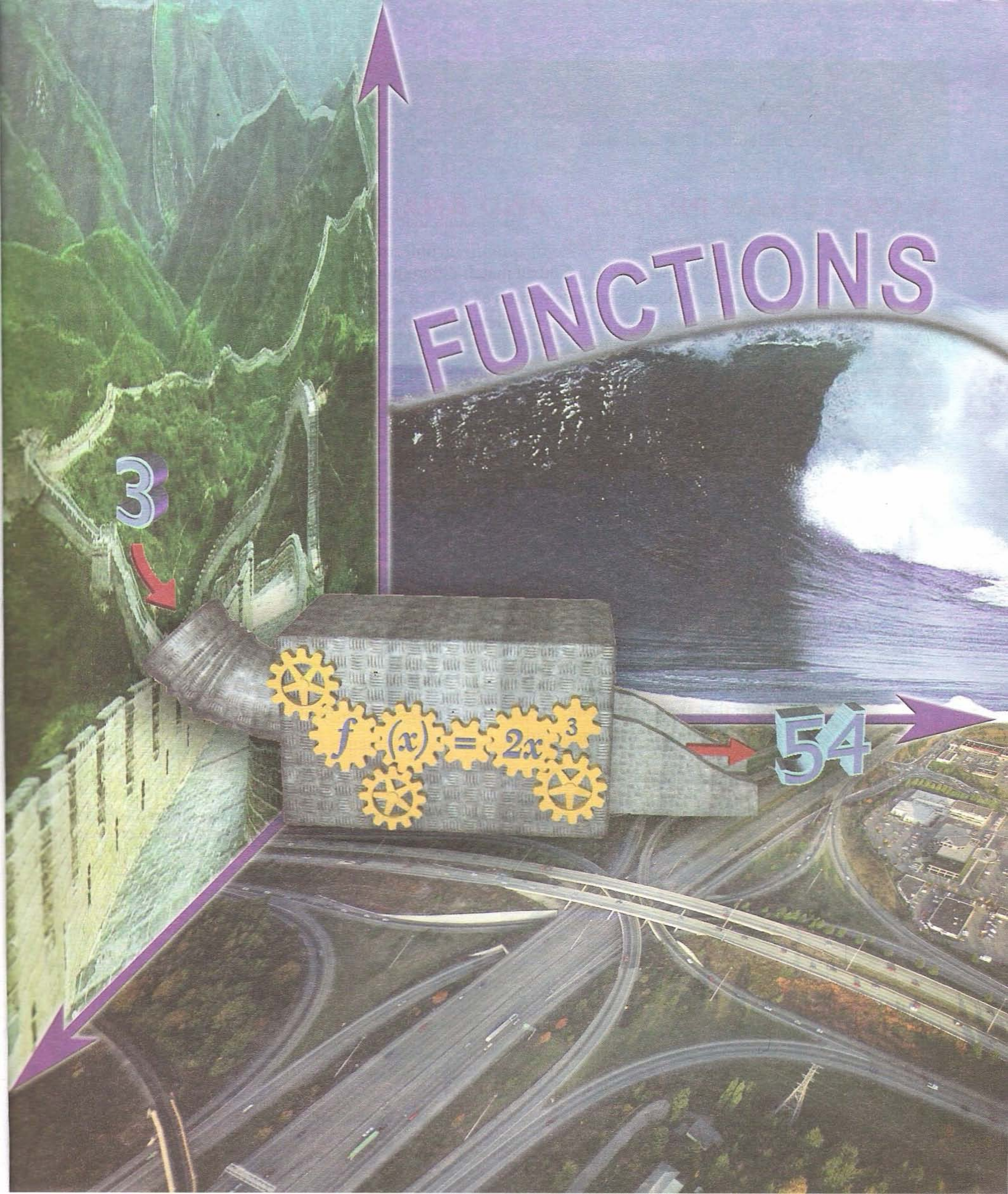
Daniel Bernoulli

During the late sixteenth century the function concepts started to take place between sets of numbers and not just quantities. Descartes wanted to reduce the solution of all algebraic problems and equations to some standard procedure and developed the idea of introducing a function analytically in 1637. This was a revolution in mathematics. The word "function" first appeared in Leibniz's manuscripts of 1694 where he used functions to stand for any quantity associated with a curve. Leibniz also introduced the words "constant", "variable", and "coordinates". Bernoulli used the word "function" in an article in 1698 on the solution to a problem involving curves. He was the first to define a function as an analytic expression. He proposed the Greek letter *phi* or *phix* be used as a notation for a function. Later Euler named a function as any equation made up of constants and variables and introduced f as a representation for function and included brackets for $f(x)$, although its origin is generally attributed to Clairaut (1734).

By the eighteenth century, the idea of a function as an analytical expression was insufficient, and a more generalized definition was developed. The definition of a function that has been used up until this century was formulated by Dirichlet. He stated that if two variables, x and y, are so related that for each value of x there corresponds exactly one value of y, then y is said to be a function of x. Today in most college algebra classes the function is defined either as a rule, or as a set of ordered pairs.



Euler introduced the function notation that we use today.


FUNCTIONS
3
$f(x) = 2x^3$
54

1 RELATIONS

A. CARTESIAN PRODUCT AND ANALYTIC PLANE

1. Ordered Pairs

Everyone knows that in a football match different scores, like 3 – 1 and 1 – 3, have a different meaning. When the score is written as 3 – 1, it is clear that the home side is the winner. But when it is written as 1 – 3, we understand that the away side is the winner. A similar approach is also valid for ordered pairs (3, 1) and (1, 3), that is, they don't have the same meaning.

Definition

ordered pair

Given that a and b are any two elements, (a, b) is called an **ordered pair** where a is the **first component** and b is the **second component**.

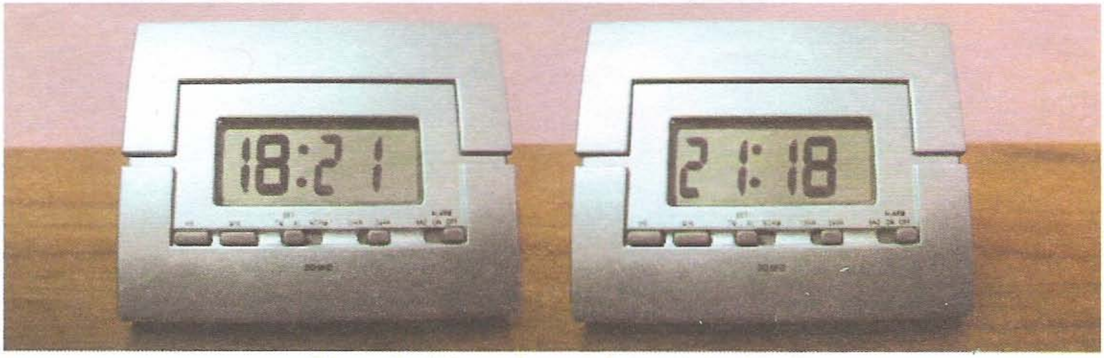
For example, below we have ordered pairs where components are numbers, names, colors, etc.:

ordered pair	*first component*	*second component*
(28, December)	28	December
(René, Descartes)	René	Descartes
(–9, 1)	–9	1
(black, white)	black	white
(170 cm, 70 kg)	170 cm	70 kg
(Descartes, René)	Descartes	René

Note that the pairs (René, Descartes) and (Descartes, René) are not the same since their components are in different order.

Note

The ordered pair (a, b) is not the same as (b, a) since they are written in different orders.



What is the difference between the two clocks above?

Example 1 Give three examples for the ordered pair (x, y) supporting the equation $2x - y = 4$.

Solution Let $x = 0$, then $y = -4$.

Let $x = 5$, then $y = 6$.

Let $x = -1.5$, then $y = -7$.

So three such ordered pairs may be $(0, -4)$, $(5, 6)$, $(-1.5, -7)$.

Here, note that we can find an infinite amount of such ordered pairs and although $(0, -4)$ is an answer, $(-4, 0)$ is not!

EQUALITY OF ORDERED PAIRS

Two ordered pairs are **equal** if, and only if, corresponding components are equal to each other. That is, $(a, b) = (c, d)$ if, and only if, $a = c$ and $b = d$.

Example 2 Given that $(2x - 1, 7) = (3, 3y - 8)$, find x and y.

Solution $(2x - 1, 7) = (3, 3y - 8)$

$2x - 1 = 3$ and $7 = 3y - 8$

$x = 2$ and $y = 5$

Note

(x_1, x_2, x_3) is named as an ordered triple.

(x_1, x_2, x_3, x_4) is named as an ordered quadruple.

$(x_1, x_2, ..., x_n)$ is named as an ordered n-tuple.

Check Yourself 1

1. How many different ordered pairs are given below?

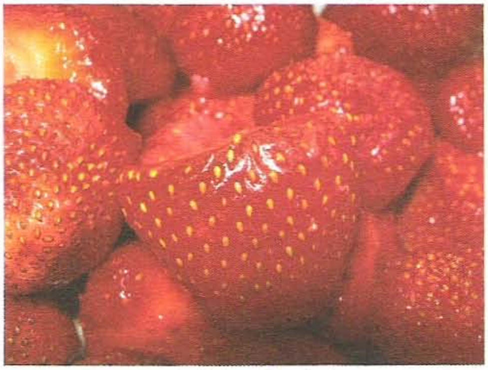
   (5 hours, 20 minutes) (28, December, 2003) (20 minutes, 5 hours) (Italy, Rome)
2. Given that $(6, 2x - y) = (x + y, -3)$, find x and y.
3. Complete the following ordered pairs (x, y) so they support $y + 1 = 3x$.

   (2, ?) (?, 0) (0, ?)

Answers

1. 3; (28, December, 2003) is an ordered triple! 2. 1; 5 3. (2, 5), (1/3, 0), (0, –1)

2. Set Notation



set of strawberries!

A set is a collection of objects. Each one of these objects that form a set is called an **element** of that set.

For example, the months of a year form a set with 12 elements. All even numbers form another set with an infinite amount of elements. All dogs with seven legs is another set with no elements. Think about the students in your class. Is it a set? How many elements does it have?

Notation

1. We usually name sets with capital letters like A, B, C, etc.
2. If a_1, a_2, a_3, ..., a_n are elements of a set A, we list all elements of this set as $A = \{a_1, a_2, a_3, ..., a_n\}$ and denote the number of elements by $n(A)$.
3. The symbol $\in$ means "is an element of". The symbol $\notin$ means "is not an element of". For example if $A = \{3, 5, 10\}$, then $3 \in A$ and $4 \notin A$.
4. The symbol $\cap$ means "intersection" and the symbol $\cup$ means "union". For example if $A = \{1, 2, 3\}$ and $B = \{2, 5\}$, then $A \cup B = \{1, 2, 3, 5\}$ since the union is the set of all elements that are either in A or B, and $A \cap B = \{2\}$ since the intersection is the set of all elements that are both in A and B.
5. To denote an empty set, that is a set with no elements, we use $\varnothing$ or $\{\ \}$.
6. The symbol $\subseteq$ means "is a subset of". For example if $B = \{-2, -1, 3, 7, 10\}$ and $C = \{-1, 3, 10\}$, then $C \subseteq B$ since each element in set C is also element of set B.

Example 3 Given that $A = \{1, 4, 5, 7\}$ and $B = \{$all odd numbers between 2 and 8$\}$,

a. find $n(A)$ and $n(B)$.

b. find $A \cup B$ and $A \cap B$.

c. is it correct that $9 \notin A \cup B$?

d. is it correct that $A \cap B = \varnothing$?

Solution We know that $A = \{1, 4, 5, 7\}$. If we list the elements of B, we get $B = \{3, 5, 7\}$.

a. $n(A) = 4$ and $n(B) = 3$.

b. $A \cup B = \{1, 3, 4, 5, 7\}$ and $A \cap B = \{5, 7\}$.

c. Clearly 9 is not an element of $A \cup B$. So $9 \notin A \cup B$ is correct.

d. Since $n(A \cap B) = 2$, $A \cap B = \varnothing$ is incorrect.

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Note

Certain letters are reserved for important number sets. These are $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$:

$\mathbb{R}$ is the set of real numbers.
$\mathbb{Q}$ is the set of rational numbers.
$\mathbb{Z}$ is the set of integers.
$\mathbb{N}$ is the set of natural numbers.

Note that all these sets contain an infinite amount of elements, so it is impossible to list them.

Notation

$A = [a, b]$ denotes the set of all real numbers between a and b making it inclusive.

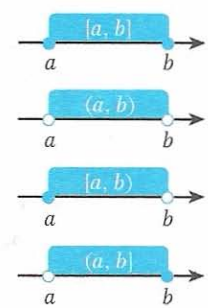
$A = (a, b)$ denotes the set of all real numbers between a and b making it exclusive.

$A = [a, b)$ denotes the set of all real numbers between a and b, where b is excluded.

$A = (a, b]$ denotes the set of all real numbers between a and b, where a is excluded.

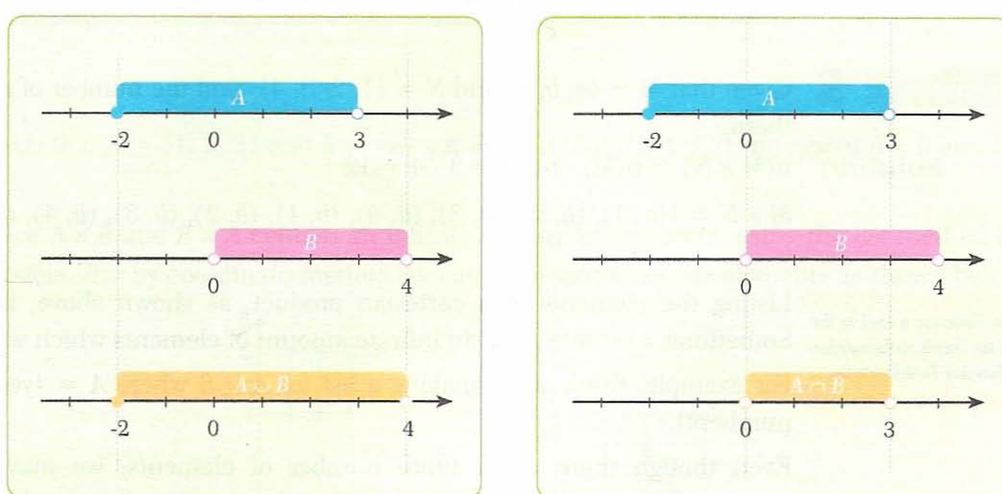
Instead of $A = [a, b]$ we can also use the notation $A = \{\text{all } x \in \mathbb{R} \text{ such that } a \le x \le b\}$ or $A = \{x \in \mathbb{R} \mid a \le x \le b\}$.

Similar notation can be used for the other sets described above.



Example 4 Given that $A = [-2, 3)$ and $B = (0, 4)$, find $A \cup B$ and $A \cap B$.

Solution



Here we cannot list all of the elements of the given sets since they contain an infinite amount of elements. $A \cup B$ contains elements that are in A or in B so $A \cup B = [-2, 4)$. $A \cap B$ contains elements that are both in A and B so $A \cap B = (0, 3)$.

Check Yourself 2

1. Given that $A = \{1, 2, 5, 9, 12\}$ and $B = \{\text{all } x \in \mathbb{N}$ that are less than 14 and divisible by 3$\}$ find $n(B)$, $A \cup B$, $A \cap B$.
2. Given that $A = [-5, 3]$ and $B = (2, 4]$, find $A \cup B$ and $A \cap B$.

Answers

1. 4, $\{1, 2, 3, 5, 6, 9, 12\}$, $\{9, 12\}$ 2. $[-5, 4]$, $(2, 3]$

3. Cartesian Product

Definition

cartesian product

Let A and B be two non-empty sets. The set of all ordered pairs, whose first component is from A and whose second component is from B, is called the **cartesian product** of A and B and is denoted by $A \times B$.

$A \times B$ is read "A cross B", not "A times B".

For example, if $A = \{1, 2, 3\}$ and $B = \{x, y\}$, then

$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$ and

$B \times A = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$.

Here, $A \times B$ and $B \times A$ are clearly different sets. The only common property between them is that they have the same number of elements, which is equal to the product of number of elements of A and B. Note that $A \times B = B \times A$ only when A and B are equal sets.

NUMBER OF ELEMENTS OF A CARTESIAN PRODUCT

Given two sets A and B, $n(A \times B) = n(B \times A) = n(A) \cdot n(B)$.

Example 5 Given that $M = \{a, b, c\}$ and $N = \{1, 2, 3, 4\}$, find the number of elements of $M \times N$ and list them.

Solution $n(M \times N) = n(M) \cdot n(N) = 3 \cdot 4 = 12$

$M \times N = \{(a, 1), (a, 2), (a, 3), (a, 4), (b, 1), (b, 2), (b, 3), (b, 4), (c, 1), (c, 2), (c, 3), (c, 4)\}$

The term *Cartesian* is used in the name of the French mathematician and philosopher René Descartes.

Listing the elements of a cartesian product, as shown above, is called the **list method**. Sometimes a set may have an infinite amount of elements which will result in an endless list. For example, think about making a list for $A \times B$ where $A = \{\text{yes, no}\}$ and $B = \{\text{all even numbers}\}$.

Even though there are a finite number of elements, we may want to see the whole representation rather than the list. For these cases the **coordinate method** is the most efficient way to show the representation. To represent all the elements of a cartesian product by the coordinate method we choose the horizontal axis for the first component and the vertical axis for the second component.

Example 6 Given that $C = \{\text{black, white}\}$ and $D = \{1, 2, 3, 4, 5, 6\}$, represent $C \times D$ by:

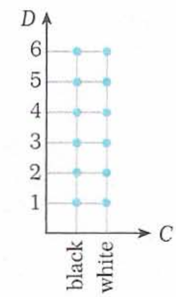
a. the list method.

b. the coordinate method.

Solution a. By the list method, we have

$C \times D = \{$(black, 1), (black, 2), (black, 3), (black, 4), (black, 5), (black, 6), (white, 1), (white, 2), (white, 3), (white, 4), (white, 5), (white, 6)$\}$.

b. By the coordinate method, we have



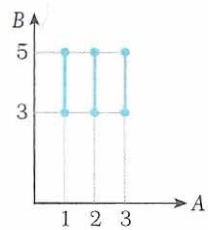
$C \times D$

Here note that the horizontal axis denotes elements of C and the vertical axis denotes elements of D. Since $n(C \times D) = 12$, we have 12 points plotted.

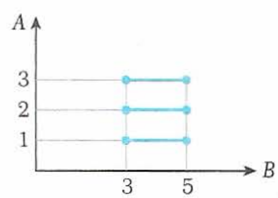
A chessboard is a cartesian product of $\{A, B, C, D, E, F, G, H\}$ and $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Example 7 Given that $A = \{1, 2, 3\}$ and $B = \{\text{all } x \in \mathbb{R} \text{ such that } 3 \le x \le 5\}$, represent $A \times B$ and $B \times A$.

Solution Since $A \times B$ and $B \times A$ contain an infinite amount of elements, using the list method is not possible. But by coordinate method we can represent all of the elements as shown below:



$A \times B$

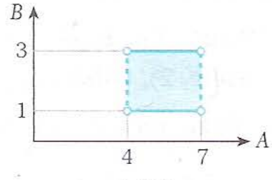


$B \times A$

How many points are plotted on each plane?

Example 8 Given that $A = (4, 7)$ and $B = [1, 3]$ represent $A \times B$.

Solution By the coordinate method we have



$A \times B$

Note that the vertical boundaries of the rectangle are not included in $A \times B$.

4. Analytic Plane

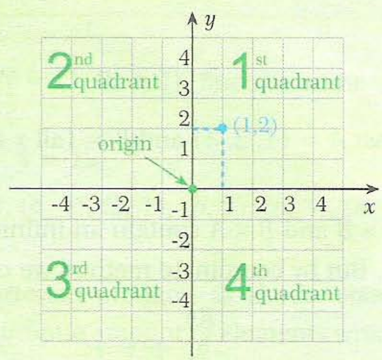
RENÉ DESCARTES
(1596-1650)

French mathematician and philosopher, René Descartes believed science and mathematics could explain and predict events in the physical world. Descartes developed the Cartesian coordinate system for graphing equations and geometric shapes. Modern maps use a grid system that can be traced back to Cartesian graphing techniques. It is said (although the story is probably a myth) that Descartes came up with the idea for his coordinate system while lying in bed and watching a fly crawl on the ceiling of his room.

To graph the cartesian products whose elements are ordered pairs of real numbers, we need a coordinate system. The **rectangular** or **cartesian coordinate system** consists of a horizontal number line, the ***x*-axis** or the **abscissa**, which we label as x, and a vertical number line, the ***y*-axis** or the **ordinate**, which we label as y.

The plane on which such a coordinate system is constructed is called an **analytic plane** or ***xy*-plane**. Axes divide the analytic plane into four parts which are called **quadrants**. The intersection point of axes is called the **origin**.

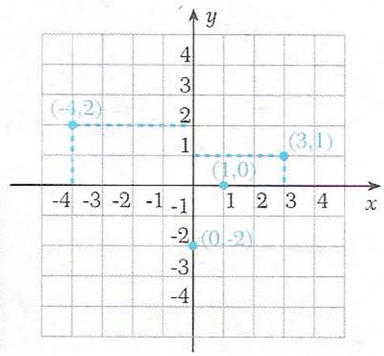
ILLUSTRATION OF ANALYTIC PLANE



Just as every real number corresponds to a point on the number line, every pair of real numbers corresponds to a point on the analytic plane. For example the pair (1, 2) corresponds to the point that lies one unit to the right of the origin and two units up. The first component, which lays on the x-axis, is 1. The second component, which lays on the y-axis, is 2. The point where the components meet is called the **plot**. Locating this point on an analytic plane is named **plotting** or **graphing** the plot.

Example 9 Plot the pairs (0, –2), (3, 1), (–4, 2), (1, 0) on an analytic plane.

Solution Note that the first component is on the x-axis and the second component lays on the y-axis.



The coordinate system that maps use stems from the Cartesian coordinate system.

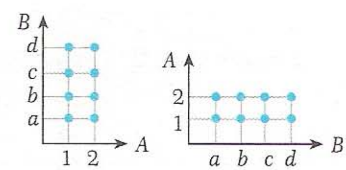
Check Yourself 3

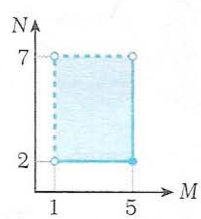
1. Given that $A = \{1, 2\}$ and $B = \{a, b, c, d\}$, represent $A \times B$ and $B \times A$ by the list method and the coordinate method.
2. Given that $M = \{\text{all } x \in \mathbb{R} \text{ such that } 1 < x \leq 5\}$ and $N = \{\text{all } x \in \mathbb{R} \text{ such that } 2 \leq x < 7\}$, express $M \times N$ by the coordinate method. Is it possible to use the list method?
3. Plot the pairs (1, –2), (0, 1), (3, 3), (–4, –1) on an analytic plane.

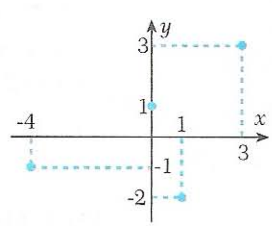
Answers

1. $\{(1, a), (1, b), (1, c), (1, d), (2, a), (2, b), (2, c), (2, d)\}$

   $\{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2), (d, 1), (d, 2)\}$



2.  ; no

3. 

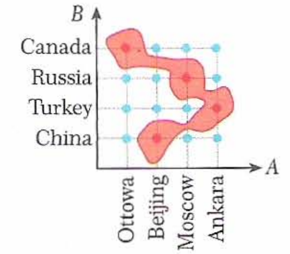
B. RELATIONS

1. Representation of a Relation

If A and B are two non-empty sets, then every non-empty subset of the cartesian product $A \times B$ is a relation defined from A to B.

For example, if A = {Ottowa, Beijing, Moscow, Ankara} and B = {China, Turkey, Russia, Canada} we can describe a relation, say β, from A to B in the following way:

β = {all pairs (x, y) such that x is the capital of y} and can be listed as follows: {(Ottowa, Canada), (Beijing, China), (Moscow, Russia), (Ankara, Turkey)}. Here β is a subset of $A \times B$. Note that we can choose any subset of $A \times B$ to find another relation. Since each subset is a collection of ordered pairs, a relation can simply be defined as the collection of ordered pairs.



Above all elements of $A \times B$ is illustrated. Red ones are elements of relation β.

Definition

relation

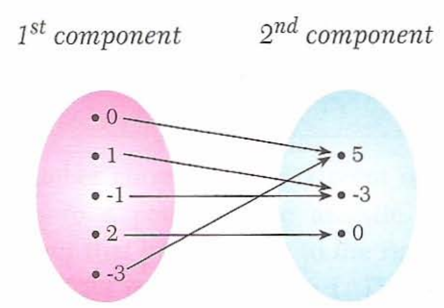
A **relation** is a set of ordered pairs. The set of all the first components is called the **domain** and the set of all the second components is called the **range** of the relation.

For example, {(0, 5), (1, –3), (–1, –3), (2, 0), (–3, 5)} is a relation since it is a set of ordered pairs. Its domain is {–3, –1, 0, 1, 2} and its range is {–3, 0, 5}.

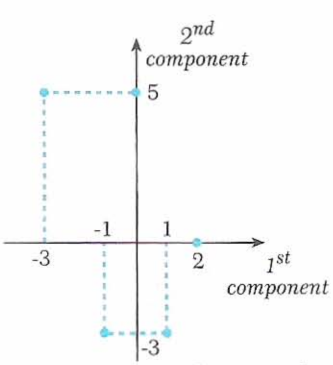
The relation above is represented by a **list**. This relation can also be represented by a **table**, by a **map**, or by a **graph** as seen below:

1st component	*2nd component*
0	5
1	–3
–1	–3
2	0
–3	5

representation by a table



representation by a map



representation by a graph

Example **10**

month	*season*
January	Winter
September	Fall
April	Spring
March	Spring
June	Summer
October	Fall
February	Winter
May	Spring

Given the following relations represented by the table above, answer the following:

a. How is the first component related with the second component?

b. Represent it by a list.

c. Represent it by a map.

d. Represent it by a graph.

e. Find the domain.

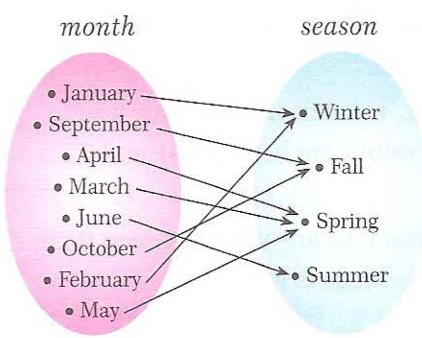
f. Find the range.

Solution Before we find the solution, to get rid of confusion, let us assume that we are on the northern hemisphere since seasons are different on the southern hemisphere of the earth.

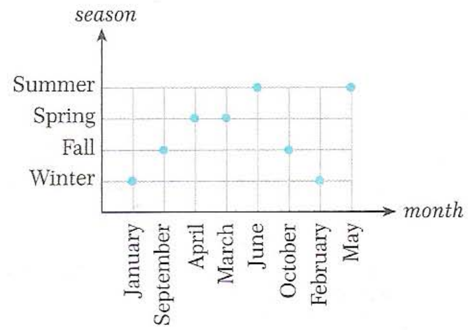
a. The first component is the month, and the second component is the season which contains the given month.

b. {(January, Winter), (September, Fall), (April, Spring), (March, Spring), (June, Summer), (October, Fall), (February, Winter), (May, Spring)}

c.



d.



e. The domain is {January, September, April, March, June, October, February, May}.

f. The range is {Winter, Fall, Spring, Summer}.

Example 11 Given the domain $\{-2, -1, 0, 1, 2\}$ and the relation containing ordered pairs of the form (x, y) such that $x < y \leq 2$ and $y \in \mathbb{Z}$,

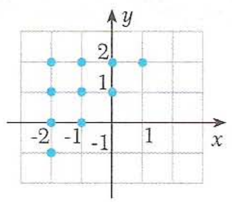
a. represent the relation by a list.

b. represent the relation by a graph.

c. find the range.

Solution a. Choose $x = -2$ from the domain. Related y values should support $-2 < y \leq 2$ and $y \in \mathbb{Z}$. So for $x = -2$, we have $y = -1, 0, 1, 2$. That gives us the elements $(-2, -1)$, $(-2, 0)$, $(-2, 1)$ and $(-2, 2)$. We proceed in the same way for $x = -1, 0, 1, 2$ and get the final list as $\{(-2, -1), (-2, 0), (-2, 1), (-2, 2), (-1, 0), (-1, 1), (-1, 2), (0, 1), (0, 2), (1, 2)\}$.

b.



c. The range is $\{-1, 0, 1, 2\}$.

How can we represent this relation by a map?

Example 12 Given the domain $[-1, 2]$ and the relation containing ordered pairs of the form (x, y) such that $y = x + 1$,

a. if possible, represent the relation by a list.

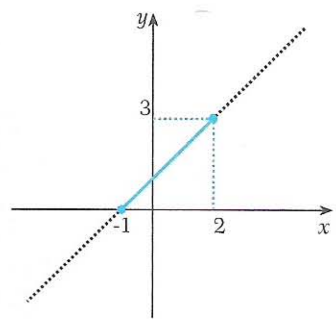
b. represent the relation by a graph.

c. find the range.

Solution a. Since the domain contains an infinite amount of elements it is impossible to list all of the elements of that relation.

b. We just sketch the line $y = x + 1$, but note that we take the part of the line whose x values are between -1 and 2 (remember the domain!).

c. As it is seen on the graph y can take any value between 0 and 3, inclusive, therefore the range is $[0, 3]$.



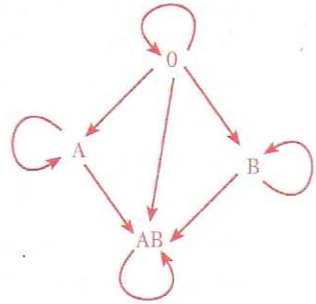
To draw a line,
1. select two numbers for x and find the y value for each.
2. plot those two points of the form (x, y).
3. connect them with a straight line and extend the line on each side.

Note

If the domain of a relation contains an infinite amount of elements we cannot represent it by a list, table or map.

Example 13 The relation between blood types is represented by the map on the right.

a. Explain the relation verbally.
b. Represent the relation by a list.
c. Find the domain.
d. Find the range.



Solution a. All blood types can give blood to the same type. Additionaly type 0 can give blood to any type and type AB can get blood from any type.

b. {(0, 0), (0, A), (0, B), (0, AB), (A, A), (A, AB), (B, B), (B, AB), (AB, AB)}

c. The domain is {0, A, B, AB}.

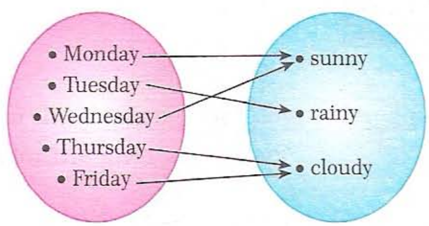
d. The range is also {0, A, B, AB}.

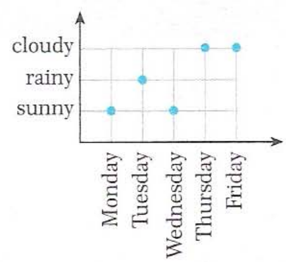
Check Yourself 4

1. Represent the relation {(Monday, sunny), (Tuesday, rainy), (Wednesday, sunny), (Thursday, cloudy), (Friday, cloudy)} by mapping and graphing.
2. Represent the relation having ordered pairs of the form (x, y) such that $y^2 = x$ in the domain {0, 1, 4} by listing and graphing.
3. Given the domain $[-3, 4]$ and the relation containing ordered pairs of the form (x, y) such that $y = -2x + 3$ represent the relation by graphing and find its range.

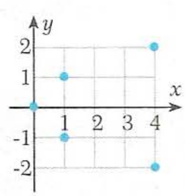
Answers

1.

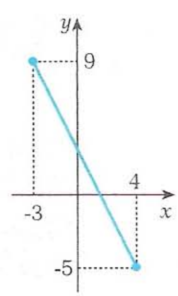




2. $\{(0, 0), (1, -1), (1, 1), (4, -2), (4, 2)\}$



3.



; $[-5, 9]$

2. Inverse of a Relation

As we learned previously, a relation is a set of ordered pairs. The inverse of a set of ordered pairs is obtained by interchanging the places of the first and the second components. Since the first and the second components are interchanged when we are finding the inverse of a relation, the domain and the range are also interchanged.

Example 14 Given the relation $\{(-2, 3), (0, 1), (-4, 0), (2, 3)\}$ containing ordered pairs of the form (x, y),

a. list the inverse relation.

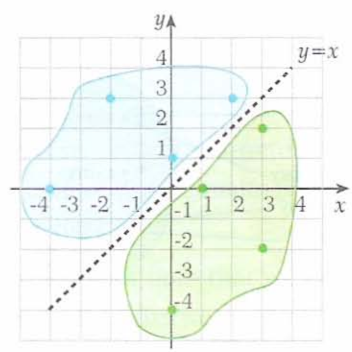
b. find the domain and range of the inverse relation.

c. graph both of these relations.

Solution a. We simply change places of components to find $\{(3, -2), (1, 0), (0, -4), (3, 2)\}$

b. The domain of an inverse relation is $\{0, 1, 3\}$ and its range is $\{-4, -2, 0, 2\}$

c. Let us plot the set of points $\{(-2, 3), (0, 1), (-4, 0), (2, 3)\}$ and $\{(3, -2), (1, 0), (0, -4), (3, 2)\}$ on the same analytic plane using different colors:

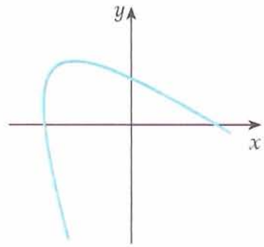


Note that each point and its inverse is symmetric with respect to the line $y = x$.

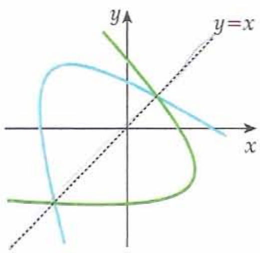
GRAPH OF INVERSE OF A RELATION

The graph of inverse of a relation is symmetric to the original graph with respect to the line $y = x$.

Example 15 Plot the graph of the inverse of the relation whose graph is given below.



Solution Since the graph of inverse of a relation is symmetry of the original graph with respect to the line $y = x$, we first draw the axis of symmetry, that is $y = x$, and then the reflection of the relation. So we get the following graph:



Note that the domain and the range of the relation and its inverse contain an infinite amount of elements.

Example 16 Given the relation containing ordered pairs of the form (x, y) such that $y = x + 2$, find the rule for the inverse relation.

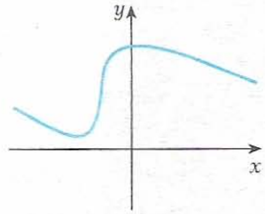
Solution If the relation was given by a list it would be enough to interchange x and y. Note that here it is impossible to list the elements of the relation since it contains an infinite amount of elements. However we can apply the same technique directly on the given rule for relation. The rule for relation is $y = x + 2$. If we interchange x and y we have $x = y + 2$. Leaving y on one side alone we get the rule for the inverse relation as $y = x - 2$.

Note

If the rule for a relation is known, then the rule for its inverse is obtained by interchanging x and y.

Check Yourself 5

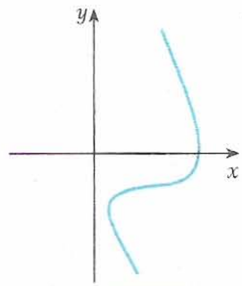
1. Given the relation $\{(1, 2), (-4, 7), (1, 0)\}$, find the inverse relation, its domain and the range.
2. Plot the graph of the inverse of the relation whose graph is given below.



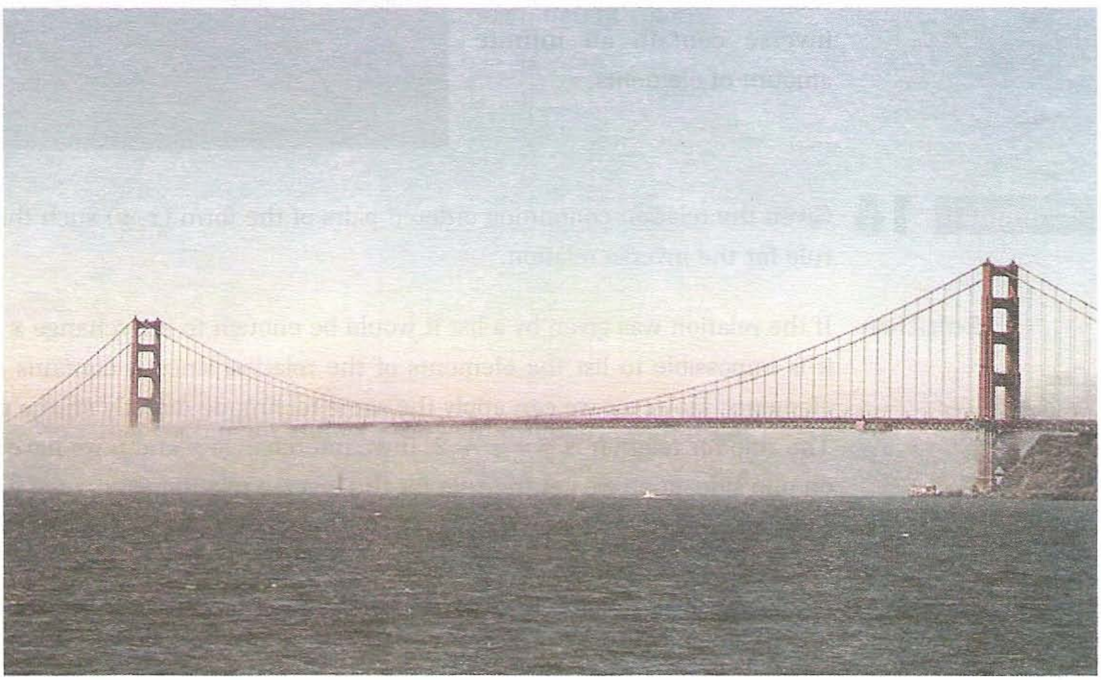
3. Given the relation containing ordered pairs of the form (x, y) such that $y = 3x - 1$, find the rule for the inverse relation.

Answers

1. $\{(2, 1), (7, -4), (0, 1)\}$; $\{0, 2, 7\}$; $\{-4, 1\}$

2. 

3. $y = \dfrac{x+1}{3}$



EXERCISES 1

A. Cartesian Product and Analytic Plane

1. Find the unknowns in the following ordered pairs.

   a. $(3x + 5, 4) = (2, y)$

   b. $(x - 2y, 3x + y) = (1, 3)$

2. Let $A = \{2, 3, 4\}$, $B = \{3, 7, 8, 9\}$, $C = \{1, 3, 5, 9, 10\}$.

   a. Find $A \cap B$.
   b. Find $n(A \cap C)$.
   c. Find $B \cup C$.
   d. Find $A \cup B \cup C$.
   e. Find $A \cap B \cap C$.
   f. Find $n((A \cup B) \cap C)$.

3. For the given sets A and B find $A \cap B$ and $A \cup B$.

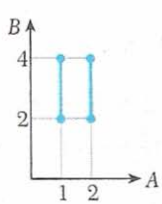
   a. $A = \{\text{all } x \in \mathbb{R} \text{ such that } -3 < x \leq 1\}$,
   $B = \{\text{all } x \in \mathbb{R} \text{ such that } -1 < x < 2\}$

   b. $A = \{\text{all } x \in \mathbb{R} \text{ such that } x > 2\}$,
   $B = \{\text{all } x \in \mathbb{R} \text{ such that } -3 \leq x < 5\}$

4. Using the following find A and B.

   a. $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

   b.



5. For the given sets A and B list $A \times A$, $B \times B$, $A \times B$, $B \times A$.

   a. $A = \{\text{all } x \in \mathbb{Z}^+ \text{ such that } x^2 < 15\}$,
   $B = \{\text{all } x \in \mathbb{Z} \text{ such that } x^2 - 1 = 0\}$

   b. $A = \{\text{months of winter in northern hemisphere}\}$,
   $B = \{2, 5, 6\}$

6. For the given sets A and B, plot $A \times B$ on an analytic plane.

   a. $A = \{1, 2, 3\}$, $B = \{4, 6\}$

   b. $A = \{\text{all } x \in \mathbb{R} \text{ such that } 1 < x < 4\}$ and
   $B = \{2, 3, 4\}$

   c. $A = \{\text{all } x \in \mathbb{R} \text{ such that } -2 < x \leq 3\}$ and
   $B = \{\text{all } x \in \mathbb{R} \text{ such that } 2 \leq x \leq 4\}$

B. Relations

7. List the following relations which have ordered pairs in the form (x, y).

   a. $y = x^2$, the domain is $\{0, 1, 2, 3\}$.

   b. $2x - 3y = 1$, the domain is $-3 < x < 4$ such that $x \in \mathbb{Z}$.

8. Represent the following relations having ordered pairs of the form (x, y) by mapping.

   a. $\{(1, b), (2, d), (3, c), (5, e)\}$

   b. $x - y = 1$, the domain is $\{1, 2, 3, 4, 5\}$.

9. Represent the following relations having ordered pairs of the form (x, y) by a graph.

a. $\{(1, 3), (2, 1), (1, 2), (2, 3)\}$

b. $y = 2x + 1$, the domain is $[0, 3]$.

c. $y = x$, the domain is $\mathbb{R}$.

d. $y \leq x - 1$, the domain is $\mathbb{Z}$.

10. Find the inverse relation, domain and range of the inverse relation for the following relations having ordered pairs in the form (x, y).

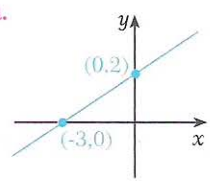
a. {(winter, December), (fall, September), (summer, June), (spring, March)}

b. $y = 2x - 3$, the domain is $[-20, 34]$

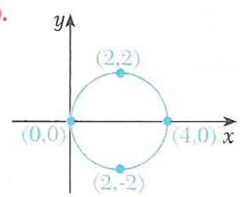
c. $y = x^2 + 2$, the domain is $[-3, 3]$

11. Graph the inverse relation for the following:

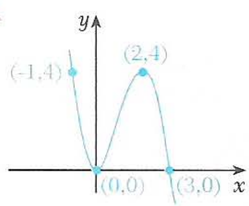
a.



b.



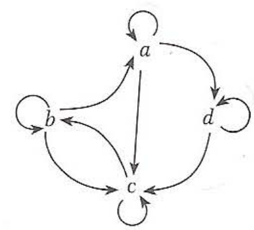
c.



Mixed Problems

12. Plot all pairs (x, y) that satisfy the equation $y = x - 1$ on an analytic plane.

13. In this figure a relation is represented by mapping. List the elements of the inverse relation.



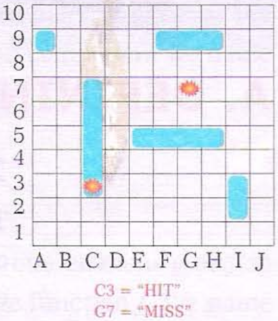
14. How many elements does $A \times B$ have if A denotes days in a week and B denotes months in a year?

15. Prove that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

16. Given that $A = \{1, 2, 3\}$, $B = \{a, b\}$, and $C = \{\text{blue, red}\}$, list $A \times B \times C$. How can you show $A \times B \times C$ by the coordinate method?

17. Given that $A = \{-1, 0, 3\}$, $B = [-5, 5]$, and $C = [1, 4)$, plot the difference of $A \times B$ from $A \times C$.

THE GAME OF BATTLESHIP



Many board games use principles of the cartesian product and the cartesian coordinate system. The Game of Battleship is a navy game where two players try to sink each other's hidden ships. On a piece of paper two players draw 10×10 tables, the first of which represent the player's ocean and the second which represents their opponent's ocean. Horizontal grids are named by letters and vertical grids are named by numbers. Each player should place five ships in his own hidden ocean as follows: "Carrier"(5 squares), "Battleship"(4 squares), "Cruiser"(3 squares), "Submarine"(2 squares), and "Destroyer"(1 square). Ships may be placed in any horizontal or vertical position - but not diagonally. Taking turns, players call out their "shots" by telling an ordered pair denoting a location in their opponent's ocean and attempting to "hit" the opponent's ships in order to sink them. When a shot is called, the opponent immediately tells whether if it is a "hit" or "miss". If the shot is a "hit", the opponent tells the player what kind of ship was hit. Players note shots of their opponent in their own view and their shots in the enemy's view to decide a further strategy of play. Strategy and some luck must be combined to be the first to locate and sink all 5 of your opponent's ships in order to win the game.

The **salvo game** version is recommended for more experienced players. It differs mainly in how many shots are taken in a turn by each player. Each player at the start takes a salvo of 5 shots in his turn. Whenever a player has had one of his ships sunk, he loses one shot for his next salvo. As his ships are removed from the game, the shots for each salvo are reduced.

The advanced salvo game offers a challenge for the expert player. It is played as salvo, except after a salvo of shots is called, the opponent simply announces how many hits were made - but not where or on what ships.

Try to remember the board games you know. Do they have any relation with the cartesian coordinate system? What can be the components of an ordered pair?

2 INTRODUCTION TO FUNCTIONS

A. DEFINING A FUNCTION

1. Definition

There are many variables around us. Some of them are closely related to each other and some of them are not. For instance, the number of students in a classroom and the quantity of oxygen in the same classroom are closely related. But the number of students in a classroom and the temperature outside are not related at all. Functions are used to show this relation between variables. Using that relation we can estimate the results for possible cases.

Below, the numbers on the right are related to the numbers on the left:

$$1 \rightarrow 1$$
$$2 \rightarrow 4$$
$$3 \rightarrow 9$$
$$4 \rightarrow 16$$

You can easily guess the rule that relates the number on the left to the number on the right. It is: "square the number". So the relation converts a number x to another number x^2. We can symbolize this as:

$$x \rightarrow x^2$$

So,

If $x = 10$, "the square of the number" is 100 *(1st sentence)*

If $x = -4$, "the square of the number" is 16 *(2nd sentence)*

If $x = 0.5$, "the square of the number" is 0.25 *(3rd sentence)*

Obviously, a way of expressing the result of the rule as shown above is not very practical or mathematical. Writing it in the form $10 \rightarrow 100$ is also not good since it is not clear which rule we are using. That is, the meaning of "$\rightarrow$" can be confusing. (it may be "add 90 to the number" or "multiply the number by 10" as well). In order to talk about this rule, in our case "square the number", we should name it as f. When we apply this rule to x, we get x^2. So f is the rule that converts x into x^2. Symbolically,

$$f(x) = x^2.$$

Let us rewrite the above sentences once more:

$f(10) = 100$ *(1st sentence)*

$f(-4) = 16$ *(2nd sentence)*

$f(0.5) = 0.25$ *(3rd sentence)*

We can write each of the numbers above on the "left" and "right" as ordered pairs like (10, 100), (–4, 16), (0.5, 0.25). Note that for each number on the "left", which is the first component, there corresponds just one number on the "right", which is the second component. We know that the mathematical relation is a set of ordered pairs. Whenever the first component of an ordered pair is associated with exactly one second component we name that relation a function.

Definition

function

A **function** f is a rule that assigns to each element x in set A exactly one element y or $f(x)$ in set B. Set A is called the **domain** and set B is called the **range** of the function f. We name x as the **independent variable** or the **argument**, and y as the **dependent variable** since the value of y depends on x.

Euler's abstract definition of function in 1755 in his Institutiones Calculi Differentialis:

If some quantities so depend on other quantities that if the latter are changed the former undergo change, then the former quantities are called functions of the latter. This denomination is of broadest nature and compromises every method by means of which one quantity could be determined by others. If, therefore, x denotes a variable quantity, then all quantities which depend on x in any way or are determined by it are called functions of it.

A function can also be thought of as a set of ordered pairs whose first components are all different. The set of all the first components of the ordered pairs is the domain of the function. The set of all the second components of the ordered pairs is the range of the function. Consider the set, A = {(cat, dog), (chicken, duck), (cat, mouse)}. The set A would not be a function because the first component, cat, is paired with 2 different second components. Consider the set $B = \{(1, 2), (2, 6), (3, 9)\}$. This is a function since each first component has only one second component paired with it. The domain of B is the set $\{1, 2, 3\}$ and the range of B is the set $\{2, 6, 9\}$. $C = \{(1, 3), (1, 4), (4, 6)\}$ would not be a function since 1 has two components paired with it, 3 and 4.

$B = \{(1, 2), (2, 6), (3, 9)\}$ FUNCTION

Different x-values

$C = \{(1, 3), (1, 4), (4, 6)\}$ NOT A FUNCTION

Same x-values

Example 17 State whether the following relations define a function or not.

a. $\{(0, 2), (0, 3), (1, 6), (2, 4), (3, 5)\}$

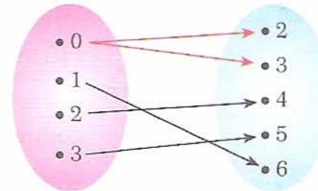
b. $\{(-3, 1), (-1, -1), (0, 1), (1, 3), (2, -2)\}$

c. A relation having ordered pairs of the form (radius, area of circle)

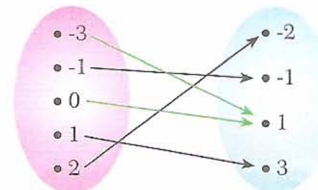
d. A relation having ordered pairs of the form (name, surname)

e. $\{(1, z), (2, d), (4, f)\}$ with the domain $\{1, 2, 3, 4\}$

Solution a. In order to have a function for each value in the domain we should have exactly one element assigned in the range. Since 0 is assigned both to 2 and 3, this is not a function.



b. The domain is $\{-3, -1, 0, 1, 2\}$. Since there is only one value assigned for each value in the domain, this is a function. Note that in the image on the right -3 and 0 are both assigned to 1, but this does not prevent it from being a function. The elements from the domain may be assigned to the same value from the range. The important point is that each element of the domain must not have more than one different value from the range.



c. For each given radius there is exactly one possible circle area. This relation is a function.

d. Two people with the same name can have different surnames. This relation is not a function.

e. As in the previous examples if nothing else is stated as the domain, the set of all the first components is the domain. For this example it would be obvious to think that the domain is $\{1, 2, 4\}$. This relation is a function if we didnt see the phrase "with domain $\{1, 2, 3, 4\}$". But the definition of a function states that every element in a domain must be assigned by exactly one element. We can see that the element "3" from the domain remains unassigned. Therefore this relation is not a function.

Note

Any function is a relation but any relation is not always a function.

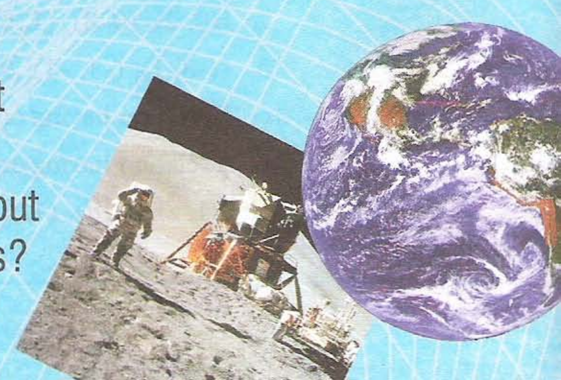
Functions around us!

To each person there corresponds just one birth date.

Have you ever heard of a person without a birth date? Or a person with two birth dates?

To each satellite there corresponds just one planet.

Have you ever heard of a satellite without a planet? Or a satellite with two planets?



Notation

If the letter f represents a function, then the notation $f(x)$ means "apply the rule f to the number x" and we read $f(x)$ as "f of x".

Example 18 Given the function with the rule "*multiply a number by 3 and add 5* ", formulize it and find its value when the number is

a. –3 b. –1 c. 0 d. 2

Solution Clearly, we have a long rule for a mathematician. If we denote the number by x and the rule by f, then we have $f(x) = 3x + 5$ and we need to find $f(-3), f(-1), f(0), f(2)$.

a. Substituting -3 in the place of x, $f(-3) = 3 \cdot (-3) + 5 = -4$.

b. Substituting -1 in the place of x, $f(-1) = 3 \cdot (-1) + 5 = 2$.

c. Substituting 0 in the place of x, $f(0) = 3 \cdot 0 + 5 = 5$.

d. Substituting 2 in the place of x, $f(2) = 3 \cdot 2 + 5 = 11$.

> $f(x)$ does not mean "f times x".

Example 19 Given the function $f(x) = |2x - 3|$,

a. find $f(1)$. b. find x, if $f(x) = 0$.

Solution a. $f(1) = |2 \cdot 1 - 3| = |-1| = 1$

b. We are looking for x value(s) for which $f(x) = 0$.

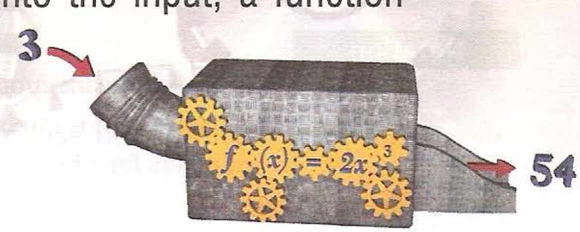
That means $|2x - 3| = 0$.

Solving this equation we get that $x = 1.5$.

> "| |" stands for absolute value.
> Absolute value of a number is
> - itself if it is nonnegative,
> - its negative if it is negative.
>
> e.g. $|25| = 25$, $|-0.7| = 0.7$

Function Machine

A common way to consider functions is through the idea of a function machine. Imagine this machine as having an input, where items are entered, and an output, where results are obtained. If a number is dropped into the input, a function machine will output a single value.



Example 20 Given the function $f(x) = x^2 - 3x$, find $f(-2) + f(4)$.

Solution To find $f(-2) + f(4)$ we use the formula $f(x) = x^2 - 3x$ twice by substituting -2 and 4 in the place of x.

In general, $f(a) + f(b) \neq f(a + b)$.

$f(-2) = (-2)^2 - 3(-2) = 10$

$f(4) = 4^2 - 3 \cdot 4 = 4$

So, $f(-2) + f(4) = 10 + 4 = 14$.

Notation

In mathematics, there are different notations used for functions. The expressions below are the most frequent notations and have the same meaning:

$$f(x) = x^2$$

$$y = x^2$$

Since this is just a matter of notation we may also use letters like g or h to denote a function and t, s, u, v to denote its argument. For the previous example if we had chosen g as the function name and u as the variable, then the formula would have been $g(u) = u^2 - 3u$ which would definitely give us the same result.

EXAMPLE 21 Given the function $g(x) = \dfrac{2x+1}{x^2-1}$, find $g(2)$, $g(2a)$, $g(a + 3)$, $g(-x)$, $g(\text{anything})$.

Solution Substituting 2 in the place of x, $g(2) = \dfrac{2 \cdot 2 + 1}{2^2 - 1} = \dfrac{5}{3}$.

Substituting $2a$ in the place of x, $g(2a) = \dfrac{2 \cdot 2a + 1}{(2a)^2 - 1} = \dfrac{4a+1}{4a^2-1}$.

Substituting $a + 3$ in the place of x, $g(a+3) = \dfrac{2 \cdot (a+3) + 1}{(a+3)^2 - 1} = \dfrac{2a+7}{a^2+6a+8}$.

Substituting $-x$ in the place of x, $g(-x) = \dfrac{2 \cdot (-x) + 1}{(-x)^2 - 1} = \dfrac{-2x+1}{x^2-1}$.

Substituting anything in the place of x, $g(\text{anything}) = \dfrac{2 \cdot (\text{anything}) + 1}{(\text{anything})^2 - 1}$.

Notation

$f : A \to B$ means function has domain A and range B.

This notation is rarely used. When it is not used, it means that the domain is all possible real numbers for which the function is defined and the range is all possible corresponding values.

Example 22 Given the function $g : \mathbb{Z} \to \mathbb{Z}$ such that $g(u) = u + 5$, find $g(0)$, $g(-2)$, $g(-0.5)$.

Solution Substituting 0 in the place of u, $g(0) = 0 + 5 = 5$.

Substituting -2 in the place of u, $g(-2) = -2 + 5 = 3$.

We cannot substitute -0.5 in the place of u since the function has integers as its domain (note that $-0.5 \notin \mathbb{Z}$), so $g(-0.5)$ is undefined.

What would you say about $g(-0.5)$ if the domain of function was $\mathbb{R}$?

Expressing a function with the help of one formula may not always be possible. Sometimes we need two, three or even more formulae to describe a function. We face these kind of situations frequently and name such functions as **piecewise defined functions**.

Example 23 Given the function $f(x) = \begin{cases} 3 - x & \text{if } x > 1 \\ x^2 + 1 & \text{if } -1 \le x \le 1, \\ 2x & \text{if } x < -1 \end{cases}$

find $f(0) + f(-4) + f(10)$.

In piecewise functions the formula is chosen according to the given value. For example we choose our way according to the destination we are going to.

Solution Note that the function is a rule. For the given function we can express the rule in words as follows:

- If the chosen value of x is more than 1, then substitute it in $3 - x$.
- If the chosen value of x is between -1 and 1 inclusive, then substitute it in $x^2 + 1$.
- If the chosen value of x is less than -1, then substitute it in $2x$.

To find $f(0) + f(-4) + f(10)$ we should calculate $f(0)$, $f(-4)$, and $f(10)$ apart from each other.

Since $-1 \le 0 \le 1$, we have $f(0) = 0^2 + 1 = 1$.

Since $-4 < 1$, we have $f(-4) = 2 \cdot (-4) = -8$.

Since $10 > 1$, we have $f(10) = 3 - 10 = -7$.

So $f(0) + f(-4) + f(10) = 1 - 8 - 7 = -14$.

Note

A piecewise function is defined so that for any value in the domain there corresponds exactly one formula.

EXAMPLE 24 Given that $f(2x + 1) = x + 7$, find $f(3)$.

Solution We cannot substitute 3 in the place of x. If we do so, we get $f(2 \cdot 3 + 1) = 3 + 7$ which gives $f(7) = 10$. But we need $f(3)$ not $f(7)$! To find it we will look for the value of x which will make the inside of the expression $f(2x + 1)$ equal to 3. Solving $2x + 1 = 3$, we find $x = 1$. Substituting 1 in the place of x, we get $f(2 \cdot 1 + 1) = 1 + 7$ which gives $f(3) = 8$.

Example 25 Given that $f(\frac{1}{x}) = x - 1$, find

a. $f(2)$. b. $f(x)$.

Solution a. We must use the same method of the previous example. To find $f(2)$ we must find the value of x which will give 2 when put in the place of the expression $\frac{1}{x}$. In other words we must solve $\frac{1}{x} = 2$, which means $x = \frac{1}{2}$. Substituting this value in the given formula we have

$$f(\frac{1}{\frac{1}{2}}) = \frac{1}{2} - 1 \text{ or } f(2) = -\frac{1}{2}.$$

b. Let $\frac{1}{x} = a$, so $x = \frac{1}{a}$. Rewriting the formula $f(\frac{1}{x}) = x - 1$ in terms of a we have $f(a) = \frac{1}{a} - 1$. Since this is just a matter of notation instead of letter "a" we can choose any other letter we like. Since we must find $f(x)$, let us use "x" in place of "a". So $f(x) = \frac{1}{x} - 1$.

Note that since we now have the formula for $f(x)$ we are able to find any value of f without repeating the procedure in part a. Try to find $f(2)$ directly using $f(x) = \frac{1}{x} - 1$. What will you get?

Check Yourself 6

1. Is the relation having ordered pairs in the form of (month, its season) a function? What about the relation with ordered pairs in the form of (season, a month of it)?
2. Given the function $f(x) = x^2 - 5x$, find $f(0)$, $f(3)$, $f(-a)$, $f(x + 1)$, $f(a) + f(b)$.
3. Given the function $f(x) = \begin{cases} x^2 - x & \text{if } x > 0 \\ 4 & \text{if } x \leq 0 \end{cases}$, find $f(-2)$, $f(0)$, $f(5)$.
4. Given that $f(2x - 3) = x^2 - x$, find $f(1)$.

Answers

1. yes, no 2. $0, -6, a^2 + 5a, x^2 - 3x - 4, a^2 - 5a + b^2 - 5b$ 3. 4, 4, 20 4. 2

2. Applied Problems

Functions are widely used for modelling real life data. In modelling functions, we must be able to translate the verbal description into mathematical language. We do this by assigning symbols to represent the independent and dependent variables and then finding the function that relates these variables. Once we model a real life situation, we are able to estimate the result for any specific value.

Example 26 The area of a rectangle is 40 m^2.

a. Express its perimeter P as a function of its width.

b. Using the required function find the perimeter of the rectangle if its width is 5 meters.

Given a rectangle with sidelength a and b
Perimeter $= 2(a + b)$
Area $= a \cdot b$

Solution a. Let x m be the width of the rectangle. Since its area is 40 m^2, its length is $\frac{40}{x}$ m. We know that the perimeter of a rectangle is twice the sum of its length and width. So the perimeter can be expressed as $P(x) = 2(x + \frac{40}{x})$.

b. Simply we need to find $P(5) = 2(5 + \frac{40}{5}) = 26$. The perimeter of the rectangle is 26 meters.

Example 27 On a certain route an airline carries 8000 passengers per month, each paying \$50. The airline wants to increase the fare. However, the market research estimates that for each \$1 increase in fare, it will lose 100 passengers. Express the airline's monthly income as a function of increase in fare.

Solution Let x be the increase in fare. Then

the new fare $= 50 + x$,

the number of passengers $= 8000 - 100x$ *(each \$1 increase is a decrease of 100 in passengers)*.

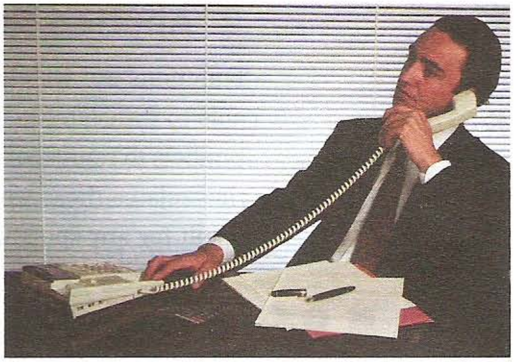
So if they increase the fare by x, the new monthly income will be

$(\textit{new fare}) \times (\textit{number of passengers}) = (50 + x)(8000 - 100x)$

Let f be the income function.

So $f(x) = (50 + x)(8000 - 100x)$ or $f(x) = -100x^2 + 3000x + 400000$.

Example 28 A telephone company does not charge any money for calls that lasts at most 15 seconds, \$0.2 per minute for calls that last at most 4 minutes and \$0.15 multiplied by unit time after 4th minute. Express the call cost as a function in terms of minutes.



Solution Let x be the call length in minutes and $f(x)$ be the cost function.

Clearly $x > 0$.

When $x \leq 0.25$ (15 seconds = 0.25 minutes), $f(x) = 0$.

When $0.25 < x \leq 4$, $f(x) = 0.2x$.

When $x > 4$, besides 0.2×4 which is the cost of first 4 minutes, the caller will pay \$0.15 multipled by extra time after the 4th minute.

So, $f(x) = 0.8 + (x - 4)0.15$.

Since we have three different cases, the cost function will be expressed piecewisely:

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq 0.25 \\ 0.2x & \text{if } 0.25 < x \leq 4 \\ 0.15x + 0.2 & \text{if } x > 4 \end{cases}$$

Check Yourself 7

1. The area of a circle is found by multiplying the square of its radius by π. Write the formula for the area of the circle function $S(r)$ where r is the radius.
2. A gardener has 140 m of fencing for her rectangular vegetable garden. Express the area as a function of one side length.

Answers

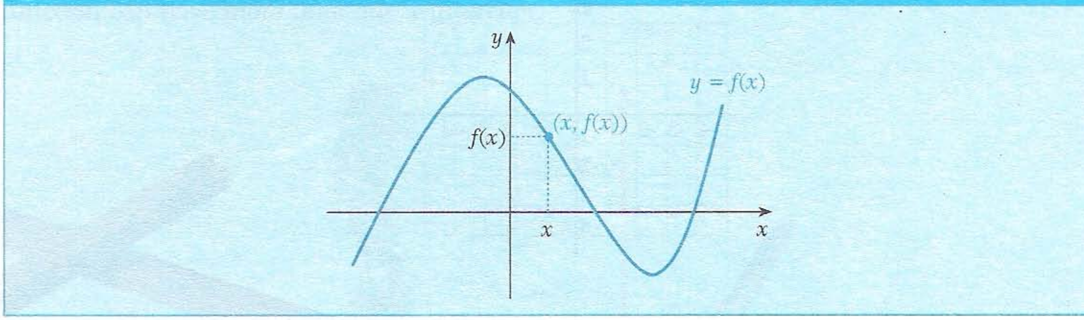
1. πr^2 2. $-x^2 + 70x$

3. Graph of a Function

a. Plotting Ordered Pairs

Visualization makes great use to get familiar with a specific function. The best way to visualize a function is to graph it. The graph of a function f is the collection of ordered pairs $(x, f(x))$ or (x, y) such that x is in the domain of the function.

ILLUSTRATION OF THE GRAPH OF A FUNCTION

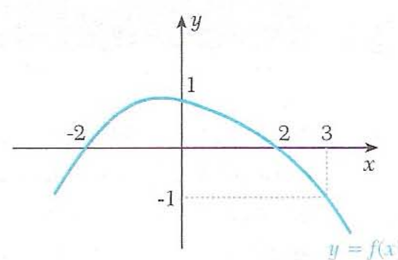


Keep in mind that x is the directed distance from the y-axis, and $f(x)$ is the directed distance from the x-axis.

Example 29 Given the graph of a function on the right,

a. find $\dfrac{f(-2)+2f(3)}{f(0)}$.

b. find x, if $f(x) = 0$.



Solution a. In this example we have no formula to use but only the graph. To find the given value we should first find each of $f(-2)$, $f(3)$ and $f(0)$.

$f(-2)$ means the y-coordinate of the point whose x-coordinate is -2. Clearly the answer is 0. With the same procedure we find that $f(3) = -1$ and $f(0) = 1$.

Therefore, $\dfrac{f(-2)+2f(3)}{f(0)} = \dfrac{0+2\cdot(-1)}{1} = -2$.

b. To find the values of x for which $f(x) = 0$, we should find all the points whose y-coordinate is 0 and then take x-coordinates of those points as an answer. Clearly, there are two such points $(-2,0)$ and $(2,0)$ so the required x values are -2 and 2.

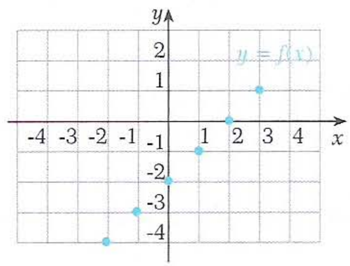
Example 30 Plot the graph of $f(x) = x - 2$ with the domain $\{-2, -1, 0, 1, 2, 3\}$.

Solution In this example the domain contains a finite number of elements. Substituting –2, –1, 0, 1, 2 in the place of x, we get

$f(-2) = -4,\ f(-1) = -3,\ f(0) = -2,\ f(1) = -1,\ f(2) = 0,\ f(3) = 1$

which gives us the ordered pairs $(-2, -4)$, $(-1, -3)$, $(0, -2)$, $(1, -1)$, $(2, 0)$, $(3, 1)$.

So we get the following graph:

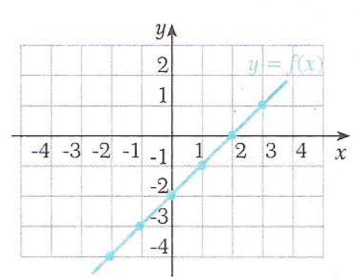


Example 31 Plot the graph of $f(x) = x - 2$.

Solution Note that the difference of this function from the one in the previous example is that its domain is any real number for which the function is defined. So we must have an infinite amount of points plotted on our graph. Let us evaluate $f(x)$ for some values of x (*see the table on the right*). This will give us the following line as graph:

x	$f(x)$
–2	–4
–1	–3
0	–2
1	–1
2	0
3	1

If domain is not given with the formula, then it is the set of all possible real numbers for which the function is defined.



Graph of $f(x) = mx + n$ is a line.

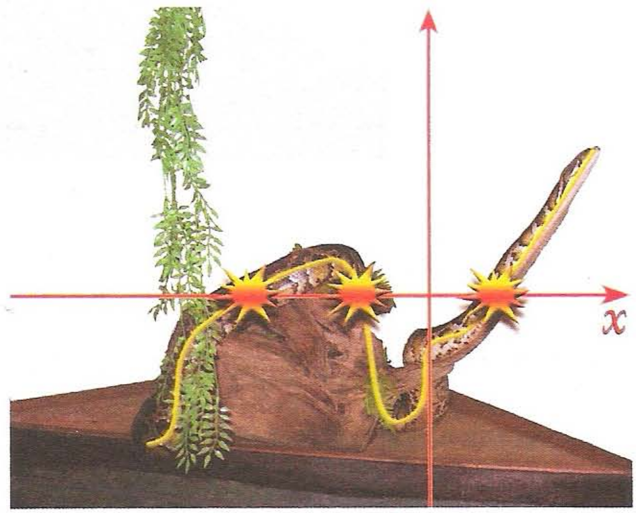
b. Intercepts of a Function

An x-intercept is the first component of the ordered pair $(a, 0)$ where a is any real number.

If the graph of a function f crosses the x-axis, then the function has an x-intercept. To find the x-intercepts, we need to find all the x-values that support the equation $f(x) = 0$. Since an equation may have more than one solution, the graph of a function can cross the x-axis more than once.

Note

The x-intercept(s) of a function are also called the **zeros** or the **roots** of the function. A function can have more than one x-intercept.



x-intercepts are on the x-axis.

Example 32 Find the zeros of the following functions:

a. $f(x) = 2x + 10$ b. $g(x) = x^2 - 9$

Solution a. Zeros of the function support the equation $f(x) = 0$ which means $2x + 10 = 0$. So at $x = -5$ function has a zero.

b. Solving $g(x) = 0$ which means $x^2 - 9 = 0$, we get $x = 3$ and $x = -3$.

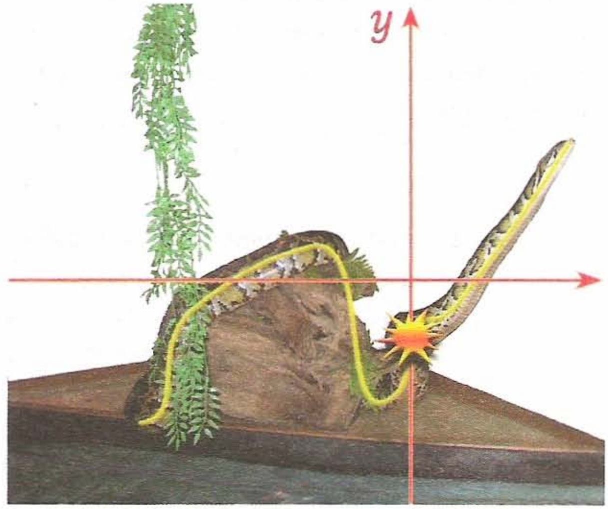
A y-intercept is the second component of the ordered pair $(0, b)$ where b is any real number.

Similarly, if the graph of a function f crosses the y-axis, then the graph of the function has a y-intercept. To find the y-intercepts, we need to let $x = 0$, and then solve for y. Since we are talking about a function, $f(0)$ can have at most one value. That means that the graph of a function can cross the y-axis at most once.

Note that we used the phrase "at most" since it is possible to have no y-intercept if 0 is not in the domain of function.

Note

A function can have at most one y-intercept.

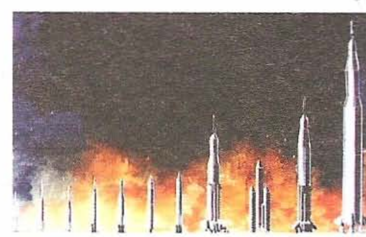


y-intercepts are on the y-axis.

EXAMPLE 33 Find the y-intercept of the function $f(x)=\dfrac{x+3}{x^2+1}$.

Solution To find the y-intercept we substitute $x = 0$. That gives us $f(0)=\dfrac{0+3}{0^2+1}=3$.

c. Vertical Line Test for Functions

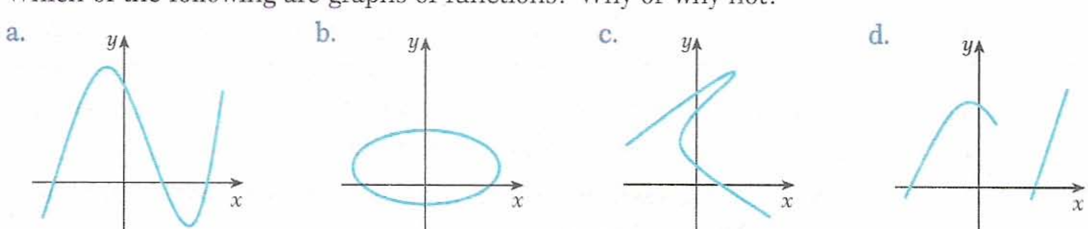


By definition, a function needs to have at most one y-value assigned to each x-value. That is, a vertical line can cross the graph of the function at most once for any x-value.

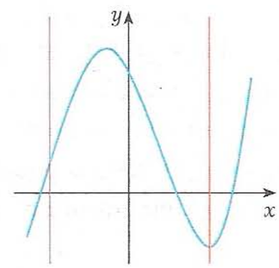
VERTICAL LINE TEST FOR FUNCTIONS

A set of points in the coordinate plane is the graph of a function if, and only if, no vertical line crosses the graph at more than one point.

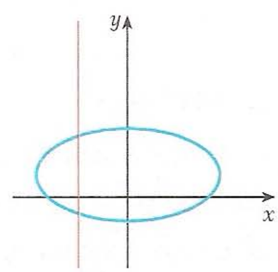
Example 34 Which of the following are graphs of functions? Why or why not?



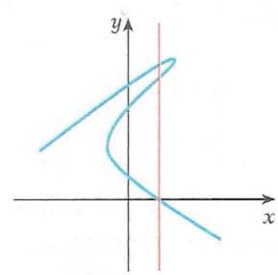
Solution a. As demonstrated below, no vertical line intersects the graph at more than one point. So the graph belongs to a function.



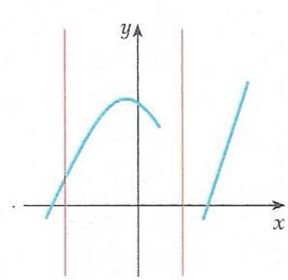
b. Since we can find a vertical line which intersects the graph at more than one point, this graph does not belong to a function.



c. Since we can find a vertical line which intersects the graph at more than one point, this graph does not belong to a function.

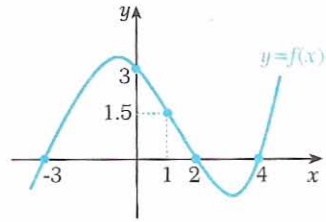


d. Since no vertical line intersects the graph at more than one point, this graph belongs to a function.



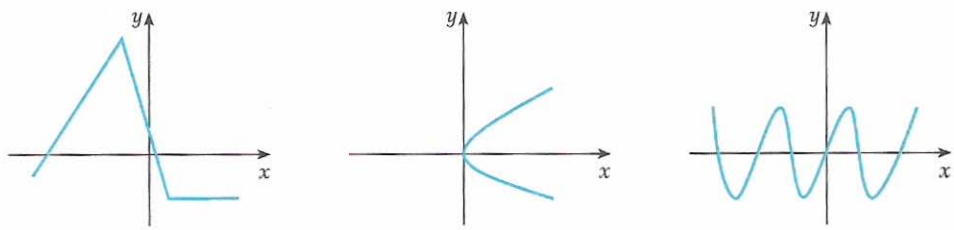
Example 35 Plot the graph of a function whose x-intercepts are -3, 2, 4 and whose y-intercept is 3 if the graph passes through the point $(1, 1.5)$.

Solution Our graph should cross the x-axis at -3, 2, 4 and the y-axis at 3. Moreover, no vertical line should intersect the graph at more than one point and the point $(1,1.5)$ should be on the graph. On the right we have a possible graph. Note that we can plot an infinite amount of different graphs using these data.



Check Yourself 8

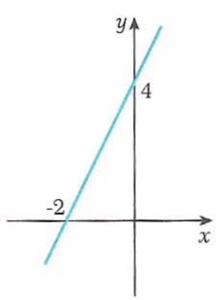
1. Plot the graph of the function $f(x) = 2x + 4$.
2. Find the x-intercept(s) of the function $f(x) = 3x - 1$.
3. Find the y-intercept of the function $g(x) = x^2 - 2x + 7$.
4. State whether the following graphs belong to functions or not. Explain your answer.



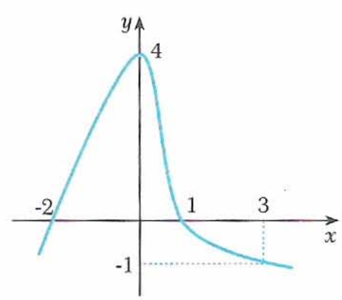
5. Plot the graph of any function with x-intercepts -2, 1 and y-intercept 4 so that the graph passes through the point $(3, -1)$.

Answers

1.



2. $\frac{1}{3}$ 3. 7 4. yes, no, yes 5.



4. Domain of a Function

You can think of candies in the jar as x values and the domain.

If a function f does not model data or come with conditions, its domain is the largest set of real numbers x for which $f(x)$ is a real number.

For example, an area of a circle with radius x can be modelled such that $f(x) = \pi x^2$. Since the radius must be positive, the domain is the set of all positive numbers.

The domain of a function may be stated explicitly. For example, if we write

$$f(x) = x + 3 \qquad x \in [2, 5),$$

then the domain is the set of all real numbers x such that $2 \leq x < 5$.

Or if we write

$$f(x) = \frac{x^2 - 4x + 3}{2x + 1} \qquad f : \mathbb{Z} \to \mathbb{R}.$$

then the domain is the set of all integers.

If the function is given by an algebraic expression and the domain is not stated explicitly, then the domain is the set of all real numbers for which the expression is defined as a real number.

For example, the function $f(x) = \sqrt{x-2}$ is defined when $x - 2 \geq 0$. So its domain is the set of all real numbers more than or equal to 2, that is $x \in [2, \infty)$.

Sometimes to find the domain of a function, we find all values that make the function undefined and throw them out of the set of real numbers.

For example, the function $f(x) = \dfrac{1}{x}$ is not defined at $x = 0$, so its domain is any real number x except 0, that is, $x \in (-\infty, 0) \cup (0, \infty)$ or $x \in \mathbb{R} \setminus \{0\}$.

Notation

To express the domain of a function $f(x)$, we write “$x \in$ ”. or “ $D(f) =$ ”.

RULES FOR FINDING THE DOMAIN OF A FUNCTION

1. The polynomial functions have any real number as their domain.
2. The rational functions have any real number except the ones that make the denominator zero as their domain.
3. Even degreed root functions (square root, fourth degree root, etc.) have any real number that makes the expression under the root non-negative as their domain.
4. Odd degreed root functions (cubic root, fifth degree root, etc.) have any real number as their domain.
5. If a function contains the combination of the above functions, then the domain is found by taking the intersection of all conditions.

Example 36 Find the domain of the following functions:

a. $f(x) = x^2 + 2x - 8$ b. $f(x) = \dfrac{1}{x+2}$ c. $f(x) = \sqrt{2x+1}$ d. $f(x) = \sqrt[3]{x-7}$

Solution a. The polynomial functions are defined for any real number x. So $D(f) = \mathbb{R}$.

b. We have a rational function denominator of which should not be equal to zero which means $x + 2 \neq 0$, that is $x \neq -2$. So $D(f) = (-\infty, -2) \cup (-2, \infty)$.

c. The square root is defined when the expression under it is nonnegative.

That means $2x + 1 \geq 0$, therefore, $x \geq -\dfrac{1}{2}$. So, $D(f) = [-\dfrac{1}{2}, \infty)$.

d. The cubic root is defined for any real number x. So $D(f) = \mathbb{R}$.

Example 37 Find the domain of the following functions:

a. $f(x) = \dfrac{1}{x^2 - 2x}$ b. $f(x) = \dfrac{1}{6-x} - \sqrt{x-5} + \sqrt[3]{x^2 - 2x}$

c. $f(x) = \sqrt[4]{(12-3x)(x^2+7)} + \dfrac{1}{x^2+2x}$ d. $f(x) = \sqrt{\dfrac{1}{3-x}} - \sqrt[4]{2x+7} + x^3 - 5x$

Solution a. Since we have a fraction, the denominator should be non-zero. That means $x^2 - 2x \neq 0$ or $x(x-2) \neq 0$. That gives $x \neq 0$ or $x \neq 2$. So $D(f) = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$.

b. $\dfrac{1}{6-x}$ means $6 - x \neq 0$. $\sqrt{x-5}$ means $x - 5 \geq 0$. $\sqrt[3]{x^2-2x}$ always gives real number no matter what x is.

To find the domain we should solve the system $\begin{cases} 6 - x \neq 0 \\ x - 5 \geq 0 \end{cases}$.

So $D(f) = [5, 6) \cup (6, \infty)$.

c. $\sqrt[4]{(12-3x)(x^2+7)}$ means $(12 - 3x)(x^2 + 7) \geq 0$. $\dfrac{1}{x^2+2x}$ means $x^2 + 2x \neq 0$.

To find the domain we should solve the system

$\begin{cases} (12-3x)(x^2+7) \geq 0 \\ x^2 + 2x \neq 0 \end{cases}$ or $\begin{cases} 12 - 3x \geq 0 \\ x(x+2) \neq 0 \end{cases}$ since $x^2 + 7$ is always positive.

So, $D(f) = (-\infty, -2) \cup (-2, 0) \cup (0, 4]$.

d. $\sqrt{\dfrac{1}{3-x}} - \sqrt[4]{2x+7} + x^3 - 5x$ gives a real number when $\dfrac{1}{3-x} \geq 0$ and $2x + 7 \geq 0$.

To find the domain it is enough to solve the system $\begin{cases} 3 - x > 0 \\ 2x + 7 \geq 0 \end{cases}$.

So, $D(f) = [-3.5, 3)$.

Example 38 Write any function by using the following domains:

a. $D(f) = (-\infty, 4]$ b. $D(f) = [1, 2)$

Solution a. Any real number x that is less than or equal to 4 will work for our function, that is $x \le 4$ or $4 - x \ge 0$. If we think $4 - x$ as inside of a square root function, this will give us the domain $D(f) = (-\infty, 4]$. So $f(x) = \sqrt{4-x}$ is such a function.

b. If we think the problem backwards $x \in [1, 2)$ should be a solution that we get from a system of inequalities $\begin{cases} x \ge 1 \\ x < 2 \end{cases}$ or $\begin{cases} x - 1 \ge 0 \\ x - 2 < 0 \end{cases}$ which can be a condition for the function

$$f(x) = \sqrt[2004]{x-1} + \frac{1}{\sqrt{x-2}}.$$

Note that these are not the only possible answers.

Check Yourself 9

1. Find the domain of the function $f(x) = \sqrt[3]{-x^2 + 2x + 8}$.
2. Find the domain of the function $g(x) = \sqrt{x-1} + \dfrac{x+4}{x^2-4}$.
3. Write any function by using the domain $D(f) = (-2, 4]$.

Answers

1. $\mathbb{R}$ 2. $[1, 2) \cup (2, \infty)$ 3. $f(x) = \dfrac{\sqrt{4-x}}{\sqrt{x+2}}$

B. PROPERTIES OF FUNCTIONS

1. Equal Functions

Example 39 Are the functions $f(x)=\dfrac{x^2-1}{x-1}$ and $g(x) = x + 1$ the same? Why?

Solution We know that $\dfrac{x^2-1}{x-1}=x+1$. But this doesn't mean that $f(x) = g(x)$. To understand this fact it is enough to see that $D(f) = (-\infty, 1) \cup (1, \infty)$ and $D(g) = \mathbb{R}$. Since $D(f) \neq D(g)$, the functions $f(x)=\dfrac{x^2-1}{x-1}$ and $g(x) = x + 1$ are not the same.

EQUALITY OF FUNCTIONS

Two functions $f(x)$ and $g(x)$ are equal if, and only if,
1. $f(x) = g(x)$,
2. $D(f) = D(g)$.

Example 40 Are the functions $f(x) = (x^{0.5})^4 - (x^{0.25})^4$ and $g(x) = x^2 - x$ equal? Why?

Solution Clearly, $f(x) = (x^{0.5})^4 - (x^{0.25})^4 = x^2 - x = g(x)$

However, note that in the original

$f(x) = (\sqrt{x})^4 - (\sqrt[4]{x})^4$, which means $D(f) = [0, \infty)$.

Since $D(g) = \mathbb{R} \neq D(f)$, functions f and g are not equal.

2. Even and Odd Functions

Let us consider the function $f(x) = x^2$. Clearly,

$$f(-2) = f(2) = 4$$

$$f(-5) = f(5) = 25.$$

It is not hard to realize that the function f gives the same value for any number and its negative in its domain. So we can say that $f(-x) = f(x)$ for any $x \in D(f)$. These kind of functions are called **even** functions.

Let us consider the function $f(x) = x^3$. Clearly,

$$f(-2) = -f(2) = -8$$

$$f(-5) = -f(5) = -125.$$

For this function we can say that $f(-x) = -f(x)$ for any $x \in D(f)$. These kind of functions are called **odd** functions.

Definition even, odd functions

A function f is **even** if, for each x in its domain, $f(-x) = f(x)$.

A function f is **odd** if, for each x in its domain, $f(-x) = -f(x)$.

Example 41 Classify whether the following functions are even or odd.

a. $f(x) = 3x^4 - 4$ b. $f(x) = 5x^3 - 2x$ c. $f(x)=\dfrac{2}{2x-7}$

d. $f(x)=\dfrac{x^2+1}{3x}$ e. $f(x)=\sqrt{x+2}$

Solution We should evaluate $f(-x)$ and decide if it is the same of the given function $f(x)$ or its negative.

a. $f(-x) = 3(-x)^4 - 4 = 3x^4 - 4 = f(x)$. This function is even.

b. $f(-x) = 5(-x)^3 - 2(-x) = -5x^3 + 2x = -(5x^3 - 2x) = -f(x)$. This function is odd.

c. $f(-x)=\dfrac{2}{2(-x)-7}=\dfrac{2}{-2x-7}$

This function is neither even nor odd since $f(-x) \neq f(x)$, nor is $f(-x) \neq -f(x)$.

d. $f(-x)=\dfrac{(-x)^2+1}{3(-x)}=\dfrac{x^2+1}{-3x}=-\dfrac{x^2+1}{3x}=-f(x)$. This function is odd.

e. $f(-x)=\sqrt{(-x)+2}=\sqrt{-x+2}$. This function is neither even nor odd.

Note

A function is even, odd or neither. Generally functions that we face are neither even nor odd.

Example 42 Classify whether the following functions are even or odd.

a. $f(x)=\dfrac{x^3-2x}{x|x|}$ b. $f(x)=\dfrac{2^x-1}{2^x+1}$ c. $f(x)=\dfrac{|x-7|}{x+1}+\dfrac{|x+7|}{x-1}$

Solution a. $f(-x)=\dfrac{(-x)^3-2(-x)}{(-x)|-x|}=\dfrac{-x^3+2x}{-x|x|}=\dfrac{-(x^3-2x)}{-x|x|}=\dfrac{x^3-2x}{x|x|}=f(x)$.

This function is even. Here note that $|-x| = |x|$ since the absolute value of a number is equal to the absolute value of its negative.

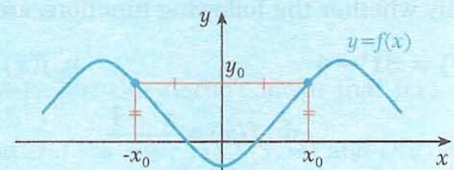
b. $f(-x)=\dfrac{2^{-x}-1}{2^{-x}+1}=\dfrac{\dfrac{1}{2^x}-1}{\dfrac{1}{2^x}+1}=\dfrac{\dfrac{1-2^x}{2^x}}{\dfrac{1+2^x}{2^x}}=\dfrac{1-2^x}{1+2^x}=-\dfrac{2^x-1}{2^x+1}=-f(x)$. This function is odd.

c. $f(-x)=\dfrac{|-x-7|}{-x+1}+\dfrac{|-x+7|}{-x-1}=\dfrac{|x+7|}{-(x-1)}+\dfrac{|x-7|}{-(x+1)}=-\left(\dfrac{|x+7|}{x-1}+\dfrac{|x-7|}{x+1}\right)=-f(x)$.

This function is odd.

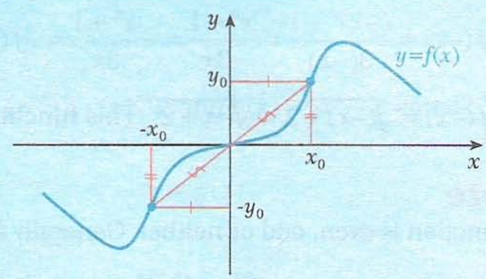
If f is even, then the points (x_0, y_0) and $(-x_0, y_0)$ are on the graph of the function f, which means the graph of an even function is symmetric with respect to the y-axis.

GRAPH OF AN EVEN FUNCTION

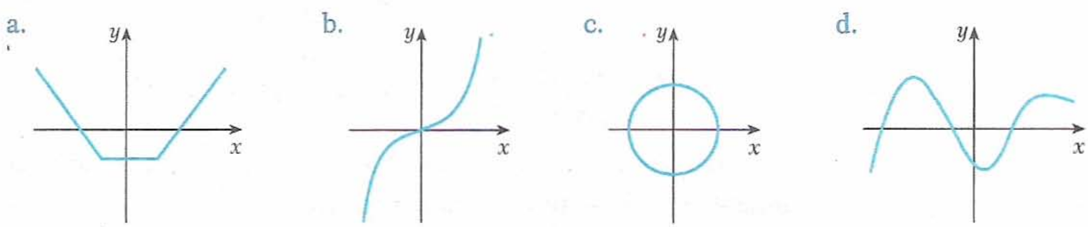


If f is odd, then the points (x_0, y_0) and $(-x_0, -y_0)$ are on the graph of the function f which means the graph of an odd function is symmetric with respect to the origin.

GRAPH OF AN ODD FUNCTION



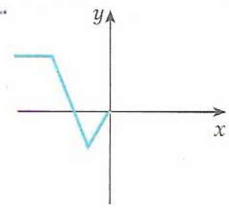
Example 43 Classify whether the following functions are even or odd:



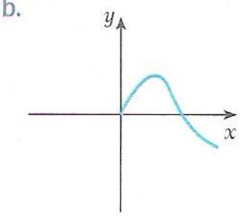
Solution
a. The graph is symmetric with respect to the y-axis making this function even.
b. The graph is symmetric with respect to the origin making this function odd.
c. Using the vertical line test we can easily state that the graph does not belong to a function. So we cannot talk about the function being even or odd.
d. The graph has no symmetry with respect to the y-axis or the origin. The function is neither even nor odd.

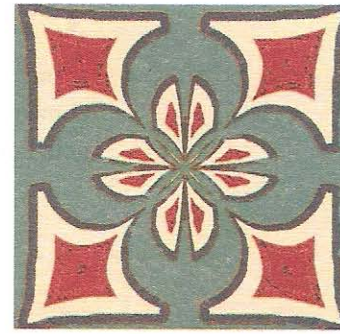
Example 44 Complete the following graphs if they belong to an even or an odd function.

a.



b.

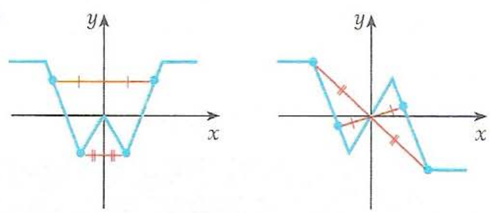




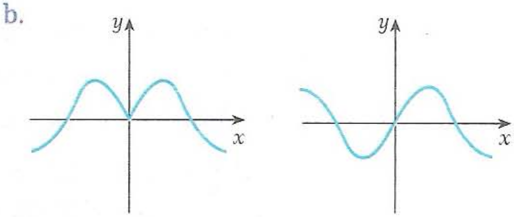
Solution In order to complete the graph for an even function just imagine the y-axis as a mirror and draw the reflection of each point on the other side.

In order to complete the graph for an odd function connect each point with the origin and extend that line to the other side of the origin until you have the same distance.

a.



b.



Check Yourself 10

1. Are the functions $f(x)=\sqrt{(x-3)(x+4)}$ and $g(x)=\sqrt{x-3}\cdot\sqrt{x+4}$ equal? Explain, why.

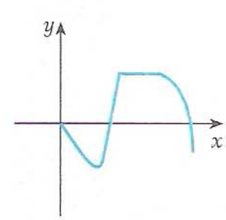
2. Classify whether the following functions are even or odd:

   $f(x) = x^7 + 4x$ $\qquad f(x)=\dfrac{x^2+2x^4}{x-4}$ $\qquad f(x) = 3^x + 3^{-x} - 2|x|$

3. Complete the graph on the right if it belongs to

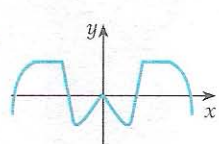
   a. an even function.

   b. an odd function.

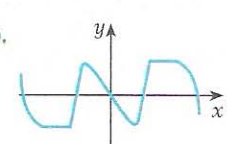


Answers

1. no 2. odd, neither, even 3. a.

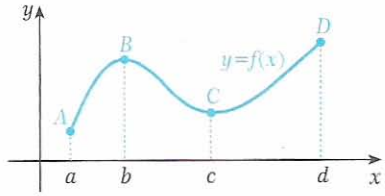


b.



3. Increasing, Decreasing and Constant Functions

It is very useful to know where the graph of a function rises and where it falls. The graph shown below rises, falls, then rises again as we move from left to right, that is as the argument gets larger.



It rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be **increasing** when its graph rises and **decreasing** when its graph falls. If there is no rise or fall in the graph, then we say the function is **constant**.

Definition

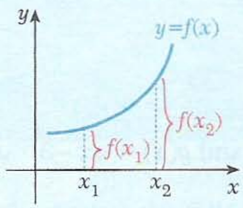
increasing, decreasing, constant function

A function f is **increasing** on an interval I if $f(x_1) < f(x_2)$ for any $x_1 < x_2$ in I.

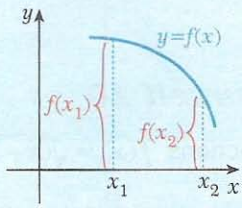
A function f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ for any $x_1 < x_2$ in I.

A function f is **constant** on an interval I if $f(x_1) = f(x_2)$ for any $x_1 < x_2$ in I.

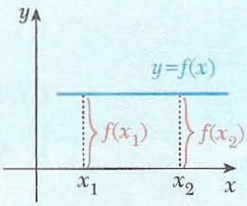
GRAPH OF INCREASING, DECREASING, CONSTANT FUNCTION



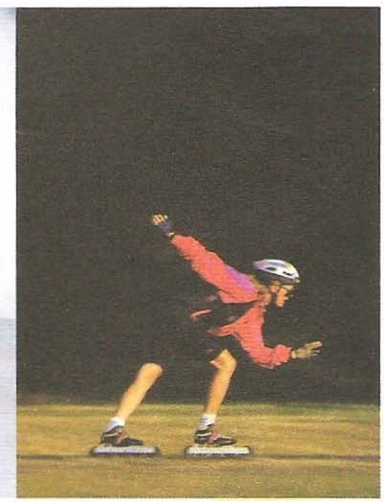
An increasing function



A decreasing function



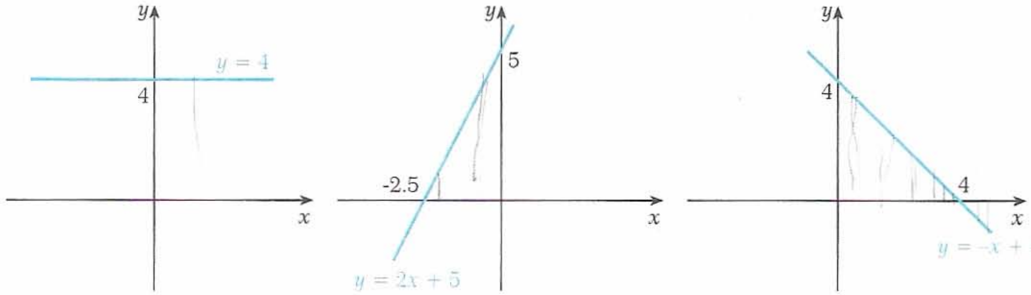
A constant function



Example 45 Investigate the following functions for increase and decrease:

a. $f(x) = 4$ b. $f(x) = 2x + 5$ c. $f(x) = -x + 4$

Solution Let us plot their graphs to see the rise and fall.



a. $f(x) = 4$ is constant on $D(f)$.

b. $f(x) = 2x + 5$ is increasing on $D(f)$.

c. $f(x) = -x + 4$ is decreasing on $D(f)$.

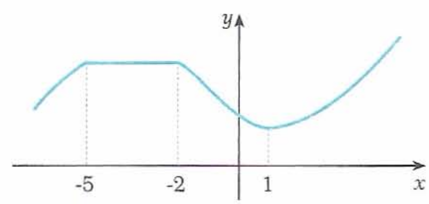
A line in the form $y = mx + n$ has m as its slope.

Note

A linear function is always

1. increasing if its slope is positive,
2. decreasing if its slope is negative,
3. constant if its slope is zero.

Example 46 State the intervals on which the function, whose graph is given on the right, is increasing, decreasing or constant.



Solution Note that when the question is about the intervals of increase and decrease we are interested in the argument, that is, the x-axis. Looking at the graph we see that there is a rise until $x = -5$ (note that since the beginning of the graph is not fixed by a point, this rise begins at minus infinity), there is a constant behaviour until $x = -2$, there is a fall until $x = 1$, and finally there is a rise "until" plus infinity. Mathematically our answer is as follows:

The function is increasing on $(-\infty, -5]$ and $[1, \infty)$, decreasing on $[-2, 1]$, constant on $[-5, -2]$. Note that we find the largest possible interval of increase and decrease. So the intervals have closed brackets whenever possible.

Example 47 Prove that $f(x)=\dfrac{5}{2x+1}$ is decreasing on $(-\infty, -0.5)$.

Solution In this problem we will use the definition. Let $x_1 < x_2 < -0.5$, then we must show that $f(x_1) > f(x_2)$ on $(-\infty, -0.5)$.

$$f(x_1)-f(x_2)=\frac{5}{2x_1+1}-\frac{5}{2x_2+1}=\frac{10(x_2-x_1)}{(2x_1+1)(2x_2+1)}$$

Since $x_1 < x_2 < -0.5$, $\begin{cases} x_2-x_1>0 \\ 2x_1+1<0 \\ 2x_2+1<0 \end{cases}$. So, $\dfrac{10(x_2-x_1)}{(2x_1+1)(2x_2+1)}>0$.

It means that $f(x_1)-f(x_2) > 0$ or $f(x_1) > f(x_2)$. Thus, $f(x)=\dfrac{5}{2x+1}$ is decreasing on $(-\infty, -0.5)$.

Example 48 Investigate $f(x) = x^2$ for increase and decrease.

Solution Note that $f(x)$ is an even function. So let's investigate it in $[0, \infty)$.

If $0 \le x_1 < x_2$, then $f(x_2) - f(x_1) = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1)$.

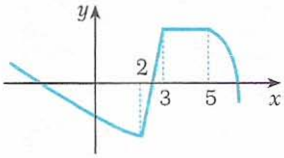
Since $0 \le x_1 < x_2$, then $\begin{cases} x_2-x_1>0 \\ x_2+x_1>0 \end{cases}$. So $(x_2 - x_1)(x_2 + x_1) > 0$.

It means that $f(x_2) - f(x_1) > 0$ or $f(x_1) < f(x_2)$.

Thus, $f(x) = x^2$ is increasing on $[0, \infty)$. Since $f(x)$ is even, it is decreasing on the other side of the y-axis (Think about the symmetry with respect to the y-axis). So the function is increasing on $[0, \infty)$, decreasing on $(-\infty, 0]$.

Check Yourself 11

1. Find a if $f(x) = (a - 3)x + 4$ is constant on $D(f)$.
2. State the intervals on which the function whose graph is given on the right is increasing, decreasing or constant.



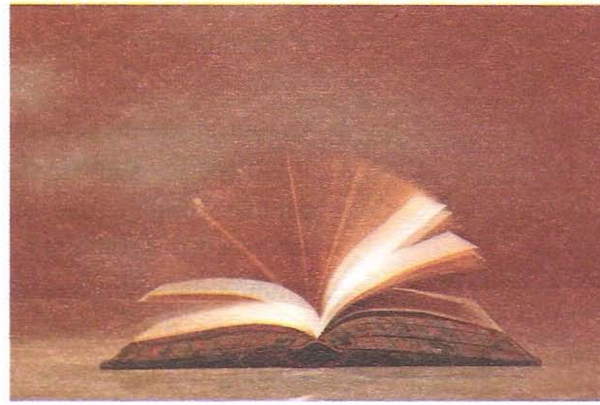
3. Prove that $f(x)=\dfrac{4}{2-x}$ is increasing on $(2, \infty)$.
4. Investigate $f(x) = x^3$ for increase and decrease.

Answers

1. 3 2. increasing on $[2, 3]$, decreasing on $(-\infty, 2]$, $[5, \infty)$, constant on $[3, 5]$
3. consider $f(x_1) - f(x_2)$ when $2 < x_1 < x_2$ 4. increasing on $\mathbb{R}$

4. Reading Graph of a Function

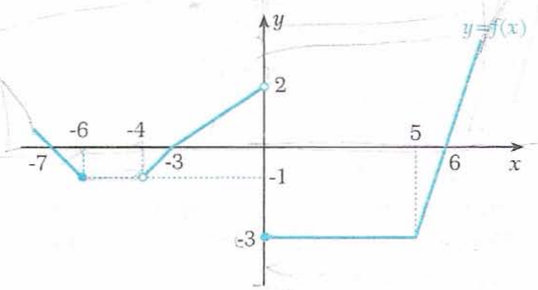
Graphs give us the opportunity to see the general details of a function at one glance. In order to get the necessary information it is important to read a graph properly. We read a graph the same way that we would read a book, from left to right. The behaviour of the function is determined by the y-values, but the intervals are reported in terms of the x-values.



Notation

1. If a point is plotted by a full circle, it means that the point is included in the graph.
2. If a point is plotted by an empty circle, it means that the point is not included in the graph.
3. If the endpoint of a graph is a full or empty circle, it means the part of the graph on that side stops at that point;otherwise it means that the graph continues up to infinity.

Example 49 Answer the following using the graph below:



a. Find $f(-6)$, $f(0)$, $f(\sqrt{3})$, $f(-5)$, $f(-4)$.

b. Find the domain and the range of f.

c. Find the x and the y-intercepts of f.

d. Solve $f(x) = -3$.

e. Solve $f(x) \geq 0$.

f. Find the intervals on which f is increasing, decreasing and constant.

g. Is f even or odd?

h. Find the minimum and the maximum value of f.

Solution a. We just find the corresponding y coordinate on the graph for each given x value, that is for -6, 0, $\sqrt{3}$, -5, -4. Clearly, $f(-6) = -1$, $f(0) = -3$, $f(\sqrt{3}) = -3$.

$f(-5)$ and $f(-4)$ are undefined since no part of the graph has x coordinate being equal to -5 or -4.

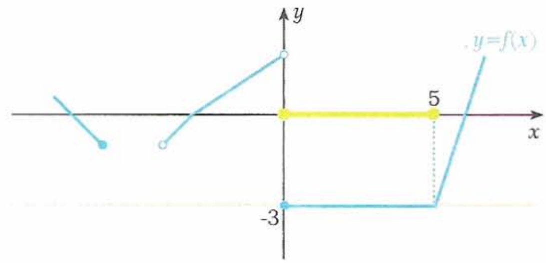
b. Graphically finding the domain means writing all of the values of x for which a vertical line intersects the graph of f. So, $D(f) = (-\infty, -6] \cup (-4, \infty)$.

Similarly finding the range means writing all of the values of y for which a horizontal line intersects the graph of f. So, $E(f) = [-3, \infty)$.

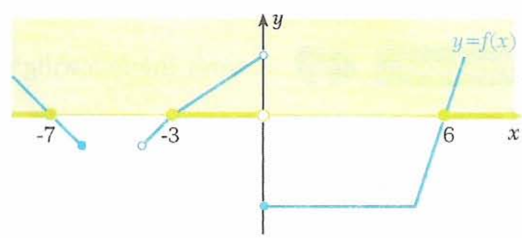
c. Finding the x-intercepts means finding the values of x for which the graph crosses the x-axis. So, the x-intercepts are -7, -3 and 6.

Finding the y-intercept means finding the value of y for which the graph crosses the y-axis. So, the y-intercept is -3.

d. To solve $f(x) = -3$ means finding values of x for which y is equal to -3. We see that there are an infinite amount of solutions: $x \in [0, 5]$.



e. To solve $f(x) \geq 0$ graphically, you must ask for which values of x is the graph of f above or at the same level with the line $y = 0$. So, the answer is $x \in (-\infty, -7] \cup [-3, 0) \cup [6, \infty)$.

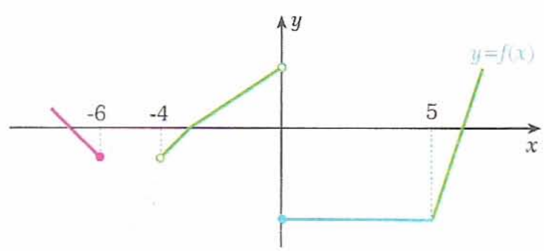


f. Recall that intervals are reported in terms of the x-values and for increase, decrease or being constant we write the largest possible interval.

f is increasing on $(-4, 0)$ and $[5, \infty)$

f is decreasing on $(-\infty, -6]$

f is constant on $[0, 5]$

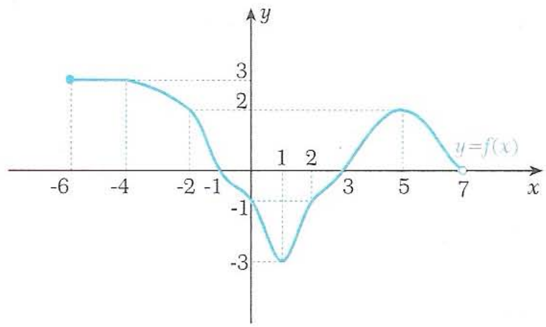


g. f is neither even nor odd since its graph has no symmetry with respect to the y-axis nor the origin.

h. The maximum value of f is the highest possible y value that the graph can reach. In our graph since it moves as high as possible we say that the maximum value doesn't exist or the maximum value is plus infinity.

The minimum value of f is the lowest possible y value that the graph can reach. In our graph the minimum value is -3.

Example 50 Answer the following using the graph below:



a. Find the domain and the range of f.

b. Find the x and the y-intercepts of f.

c. Solve $f(x) = -1$.

d. Solve $-3 < f(x) \leq -1$.

e. Solve $f(x) > 2$.

f. Solve $f(x) \geq 2$.

g. Find the intervals on which f is increasing, decreasing and constant.

h. Is f even or odd?

i. Find the minimum and the maximum value of f.

j. When does f have its minimum and maximum value?

Solution

a. $D(f) = [-6, 7)$ and $E(f) = [-3, 3]$.

b. The x-intercepts are -1, 3 and the y-intercept is -1.

c. $x \in \{0, 2\}$.

d. $x \in [0, 1) \cup (1, 2]$.

e. $x \in [-6, -2)$.

f. $x \in [-6, -2] \cup \{5\}$.

g. f is increasing on $[1, 5]$, decreasing on $[-4, 1]$ and $[5, 7)$, constant on $[-6, -4]$.

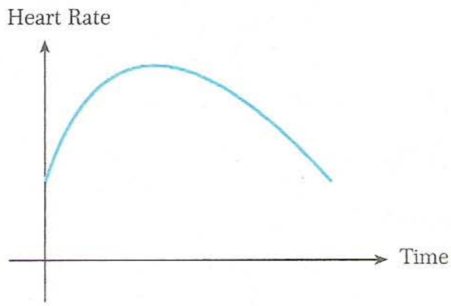
h. f is neither even nor odd.

i. The maximum value of f is 3 and the minimum value is -3.

j. f has its maximum value when $x \in [-6, -4]$ and has its minimum value when $x = 1$.

Graphs are widely used in many areas like statistics, economics, engineering, medicine, meteorology, etc. Interpreting information from them in real situations is the same as reading a graph. In fact those graphs belong to the functions that are the models of daily situations.

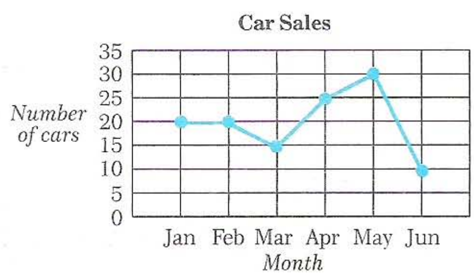
Example 51 Describe a situation that can be modelled by the graph shown.



Solution The graph shows the heart beat rate of a person in a time interval. It increases for a period of time and then decreases back to its initial condition. The person may have dealt with a sport activity during the interval of increase and after finishing that activity, the heart beat rate decreased to its normal condition.

Example 52 Answer the following questions using the graph below:

a. In what period did the car sales occur?
b. How many cars were sold in April?
c. In which month were 15 cars sold?
d. In which month were the sales the best?
e. In which month were the sales the worst?
f. In which two months were the sales same?
g. In which months did the sales increase?
h. In which months did the sales decrease?

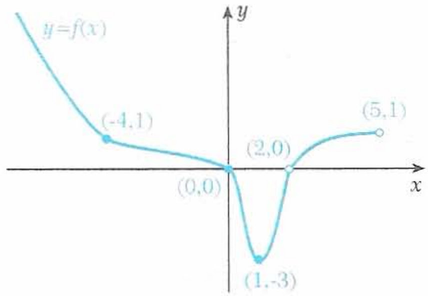


Solution a. The first six months of the year, that is from January to June, inclusive.

b. 25
c. March
d. May
e. June
f. January and February
g. April and May
h. March and June

Check Yourself 12

Answer the following using the graph below:



1. Find the domain and the range of f.
2. Find the x the and the y intercepts.
3. Find the intervals on which f is increasing and decreasing.
4. Solve $f(x) \le 1$.

Answers

1. $D(f) = (-\infty, 2) \cup (2, 5)$, $E(f) = [-3, \infty)$
2. 0;0
3. increasing on $(1, 2)$ and $(2, 5)$, decreasing on $(-\infty, 1)$
4. $x \in [-4, 2) \cup (2, 5)$

EXERCISES 2

A. Defining a Function

1. State whether the following relations are functions or not.
   a. $\{(1, x), (2, y), (3, x)\}$ with the domain $\{1, 2, 3\}$
   b. $\{(a, b), (a, c), (b, a), (c, a)\}$ with the domain $\{a, b, c\}$
   c. $\{(1, 1), (2, 2), (3, 1), (4, 1)\}$ with the domain $\{1, 2, 3, 4, 5\}$

2. Express the following rules in a function notation.
   a. Multiply by 5, then add 2.
   b. Square the number then subtract twice the number.
   c. Divide by two, then add three times its cube.

3. Express the following rules in words.
   a. $f(x) = 2x - 4$ b. $g(x)=\sqrt{x+1}$
   c. $h(x)=\dfrac{x}{3x+1}$

4. Find the required values for the following functions.
   a. If $f(x) = 2x^2 - 5x$, find
   $f(1), f(-3), f(-x), -f(x), f(\sqrt{x}), f(x + h)$.
   b. If $g(x)=\dfrac{3x-1}{x^2+2}$, find
   $g(0), g(4), g(3x), 2g(-x), g(x^2), g(2a + b)$.
   c. If $h(u)=\dfrac{u^2+1}{u^3+2u}$, find
   $h(-1), h(0), h(2u), h(x + 1), h(u^2), h^2(u)$.
   (notation $h^2(u)$ is used in the place of $(h(u))^2$ which means the square of the function.)

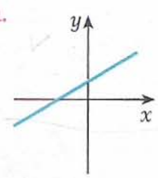
5. Given that $f(x)=\begin{cases} 2x+1 & \text{if } x>1 \\ 3 & \text{if } x=1, \\ 3x & \text{if } x<1 \end{cases}$

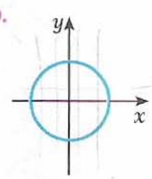
   find $\dfrac{f(2)+f(1)}{f(-1)}$.

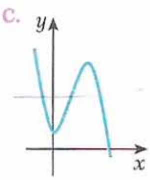
6. Plot the graphs of the following functions within the given domain.
   a. $f(x) = -x^2$, $D(f) = \{-2, -1, 0, 1, 2\}$
   b. $g(x) = 2 - x$, $D(g) = \{-1, 0, 1, 2, 3\}$
   c. $h(x) = \sqrt{x}$, $D(h) = \{0, 1, 4, 9\}$
   d. $f(x) = 2x + 1$, $D(f) = \mathbb{R}$

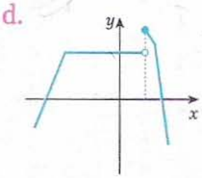
7. Find the x and the y intercepts of the following functions.
   a. $f(x) = 3x - 7$
   b. $f(x) = (x - 6)(x + 1)$
   c. $f(x)=\dfrac{2x-4}{x+5}$
   d. $f(x)=\sqrt{x+4}-5$

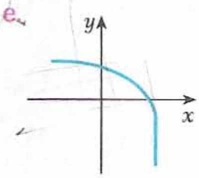
8. State whether the following graphs belong to a function or not.

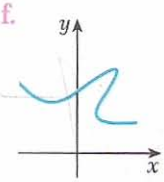
a.


b.


c.


d.


e.


f.


9. Find the domain of the following functions.

a. $f(x) = 2x + 1$

b. $f(x) = x^2 - 2x + 5$

c. $f(x) = \dfrac{x+1}{x-4}$

d. $f(x) = \dfrac{x}{x^2-9}$

e. $f(x) = \sqrt{3x-6}$

f. $f(x) = \dfrac{1}{x} - \dfrac{2}{x+3}$

g. $f(x) = \dfrac{1}{6-x} - \sqrt{x-5}$

h. $f(x) = \sqrt{x-5} + \sqrt{30-3x}$

i. $f(x) = \dfrac{1}{\sqrt{5x-1}} + \sqrt{1-x}$

j. $f(a) = \dfrac{1}{a^3-1} + \sqrt{1+a} - \sqrt{3-2a}$

k. $f(u) = \dfrac{3u-4}{u-2} + \dfrac{1}{3u-10} + \sqrt{u-3}$

10. Find the domain of the following functions.

a. $f(x) = \sqrt{15-2x-8x^2}$

b. $f(x) = \dfrac{1}{x^2-x-2} + \sqrt{\dfrac{x+4}{5-3x}}$

c. $f(x) = \dfrac{x+3}{\sqrt{x^3-x^2-x+1}}$

d. $f(x) = \dfrac{4x+3}{\sqrt{x^2-2x-8}}$

e. $f(x) = \sqrt{x^2+6x+5} + \sqrt{-x^2+x+2}$

f. $f(x) = \sqrt{x^2(1-x)(x-2)}$

g. $f(x) = \dfrac{1}{\sqrt{(x-1)^2(3x-x^2)}}$

h. $f(x) = \dfrac{\sqrt{5-|x|}}{\sqrt{|x|-1}}$

i. $f(x) = \sqrt{\dfrac{1}{2} - \left|\dfrac{3}{5-x}\right|}$

j. $f(x) = \sqrt{\dfrac{x^2-7|x|+10}{-x^2+6x-9}}$

11. Write a formula for the function by using the following domains.

a. $[3, 5)$

b. $[-4, -2) \cup (-2, 0)$

c. $[-2, 1] \cup \{3\}$

12. For which values of x are the following functions equal?

a. $f(x) = 2x - 3$ and $g(x) = 5x + 1$

b. $f(x) = x + 22$ and $g(x) = x^2 + x + 6$

c. $f(x) = \dfrac{3x-1}{x-2}$ and $g(x) = 3$

B. Properties of Functions

13. Classify whether the following functions are even, odd or neither.

a. $f(x) = x^3 + x$

b. $f(x) = x^5 + x^3 - 1$

c. $f(x) = 5 - x^2 + x^4$

d. $f(x) = \dfrac{x-1}{x^2+1}$

e. $f(x) = \dfrac{1}{x^4-3}$

f. $f(x) = \dfrac{|x|}{1+x^2}$

g. $f(x) = \dfrac{x^3-2x}{x^2+1}$

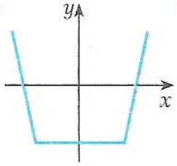
h. $f(x) = \dfrac{10^x-10^{-x}}{10^x+10^{-x}}$

i. $f(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ -x-2 & \text{if } x < 0 \end{cases}$

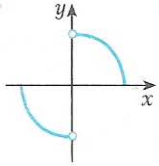
j. $f(x) = \begin{cases} 3x^2+1 & \text{if } x > 0 \\ -3x^2-1 & \text{if } x < 0 \end{cases}$

14. Given their graphs classify whether the following functions are even, odd or neither.

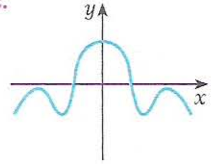
a.



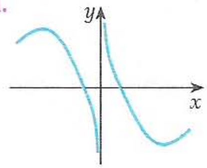
b.



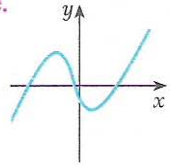
c.



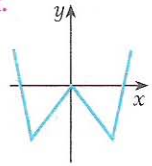
d.



e.

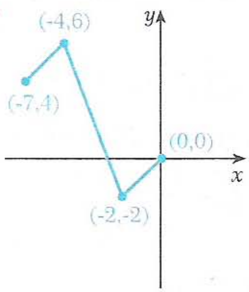


f.

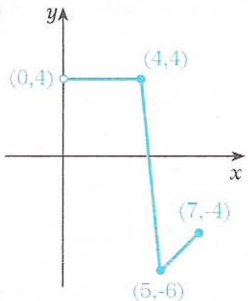


15. Complete the following graphs to get an even function and to get an odd function.

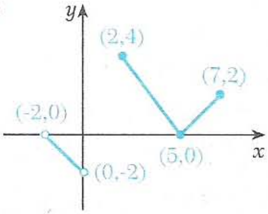
a.



b.



c.



16. Plot the graph of a function by the following intervals of increase and decrease.

a. Increasing on $(-\infty, 2]$ and $[5, \infty)$, constant on $[2, 5]$

b. Increasing on $[-5, 2]$ and $[6, 9]$, decreasing on $[2, 6]$ and $[9, 11]$

c. Increasing on $(-\infty, 3]$ and $[7, \infty)$, decreasing on $[4, 5]$, constant on $[3, 4]$ and $[5, 7]$

17. Prove that:

a. $f(x) = 3 - 2x$ is decreasing on $\mathbb{R}$.

b. $f(x) = \dfrac{4}{2-x}$ is increasing on $(2; \infty)$.

c. $f(x) = \dfrac{21x-9}{3x-1}$ is increasing on $(-\infty; \frac{1}{3})$.

d. $f(x) = \dfrac{4x+31}{x+7}$ is decreasing on $(-7; \infty)$.

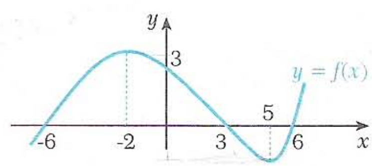
e. $f(x) = x^4 - 8x$ is increasing on $[2; \infty)$.

18. Given that $a \cdot f(x) = x \cdot f(x) + bx + 2$ and $g(x) = x + 5$, find $g(ab)$ if f is a constant function.

19. $f(x) = ax^3 + bx + 1$, $a \neq 0$, $b \neq 0$ and $f(10) = 4$, find $f(-10)$.

$\frac{b+2}{a}$

20. Answer the following using the graph below:



a. Find the domain and the range.

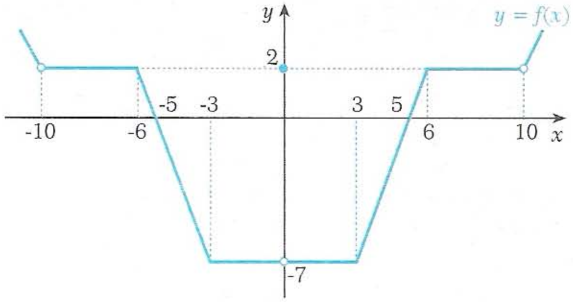
b. Find the intervals on which the function is increasing and decreasing.

c. Find the x and the y intercepts.

d. Solve $f(x - 2) = 0$.

e. Solve $f(x) < 0$.

21. Answer the following using the graph below:



a. Find the domain and range of $f(x)$.

b. Solve $f(x) = 0$.

c. Find $f(0)$.

d. Find $f(-10)$.

e. Solve $f(x) \le 0$.

f. Solve $f(x) > 2$.

g. Solve $f(x) = 2$.

h. Is $f(x)$ even or odd?

i. Write down the intervals where $f(x)$ is increasing, decreasing or constant.

j. Find the minimum and the maximum value of $f(x)$.

k. Find all the x values for which $f(x) = -7$.

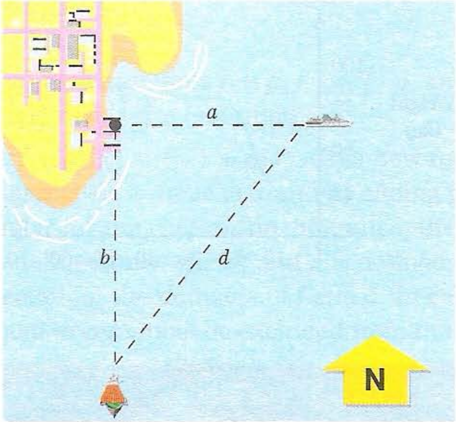
l. Find all the x values for which $f(x) \in (-7;2)$.

Mixed Problems

22. Given that $f(x) = \begin{cases} x-1 & \text{if } x > 3 \\ x^2 - 2x - 3 & \text{if } 1 < x \le 3 \\ x+4 & \text{if } x \le 1 \end{cases}$

find $\underbrace{f(f \ldots f(0) \ldots)}_{\text{2004 times}}$.

23. Two ships leave the port at the same time, one sailing east at a rate of 9 km/h and the other sailing south at 12 km/h. If t is the time (in hours) after their departure, express the distance d between the ships as a function of t.

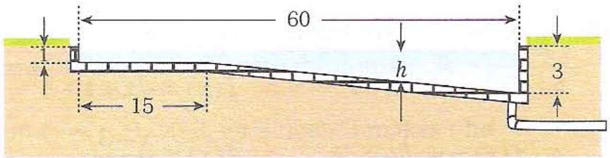


24. The freezing point of water is 0°C, or 32°F, and the boiling point is 100°C, or 212°F.

a. Express the Fahrenheit temperature F as a linear function of the Celsius temperature C.

b. What temperature increase in F corresponds to an increase in temperature of 1°C?

25. A cross section of a rectangular pool with the dimensions 60 m by 20 m is shown in the figure. The pool is being filled with water at a rate of 10 m^3 / min.

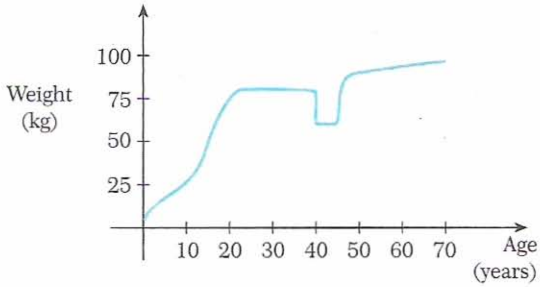


a. Express the volume V of the water in the pool as a function of time t.

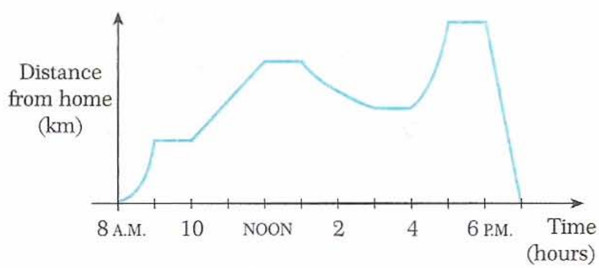
b. Express V as a piecewise function of the depth h at the deep end for $0 \le h \le 2$ and then for $2 < h \le 3$

c. Express h as a piecewise function of t.

26. The graph below gives the weight of a certain person as a function of age. Describe in words how this person's weight has varied over time. What do you think happened when this person was 40 years old?



27. The graph gives a salesman's distance from his home as a function of time on a certain day. Describe in words what the graph indicates about his travels on this day.



28. Find the range of the following functions.

a. $f(x) = 4 - x^2$

b. $f(x) = \sqrt{x+1} + 3$

c. $f(x) = \dfrac{1}{x^2+6}$

d. $f(x) = \dfrac{x-3}{x+4}$

29. If $f(x) + f(x+1) = 2x + 3$ and $f(2) = 3$, find $f(99)$.

30. If $f\left(\dfrac{x+1}{x-1}\right) = x$, find $f(-1), f(3), f(0)$.

31. If $f(x) = 2x^9 - 3x^7 + ax - 3$ and $f(11) = 2$, find $f(-11)$.

32. Find the intervals on which the function $f(x) = x^2 + |5x+1| + 5$ is increasing and decreasing.

33. If $f\left(x - \dfrac{1}{x}\right) = x^3 - \dfrac{1}{x^3}$, find $f(-x)$.

34. If $f(x) = ax^2 + bx + c$ and $f(x-1) + f(x) + f(x+1) = x^2 + 1$, find $f(2)$.

35. If $f: \mathbb{Z}^+ \to \mathbb{R}$, $f(25) = 1$, $f(x) = \dfrac{2 \cdot f(x+1) + 1}{2}$, find $f(3)$.

36. If $g(x) + 2 \cdot g\left(\dfrac{1}{x}\right) = x$, find $g(x)$ in terms of x.

3 OPERATIONS ON FUNCTIONS

A. BASIC OPERATIONS

Two functions f and g can be combined to form new functions $f + g$, $f - g$, $f \cdot g$, and f / g in a similar way that we add, subtract, multiply, and divide the real numbers.

For example, we define the function $f + g$ as $(f + g)(x) = f(x) + g(x)$. This new function is called as a sum of functions f and g. Its value at a given x value is found by adding the value of f and g at that x value. The operation is similar for the case of the difference, the product and the quotient. Clearly, the new function is defined only when f and g is defined, that the domain of the new function is also the intersection of the domains of f and g. In case of the quotient the values that make denominator equal to zero must be excluded from the domain.

SUM, DIFFERENCE, PRODUCT AND QUOTIENT OF FUNCTIONS

Let f and g be two functions with domains A and B respectively. Then

Operation	Definition	Domain
Addition	$(f + g)(x) = f(x) + g(x)$	$A \cap B$
Subtraction	$(f - g)(x) = f(x) - g(x)$	$A \cap B$
Multiplication	$(fg)(x) = f(x)g(x)$	$A \cap B$
Division	$(f/g)(x) = f(x)/g(x)$	$A \cap B$, $g(x) \neq 0$

Example 53 Given that $f(x) = x^2 - 4$ and $g(x) = x + 3$, find

a. $f + g, f - g, fg, f / g$ and their domains.

b. $(f + g)(1)$, $(f - g)(1)$, $(fg)(1)$, $(f / g)(1)$.

Solution a. Note that $D(f) = \mathbb{R}$ and $D(g) = \mathbb{R}$. So we have

$(f+g)(x) = f(x) + g(x) = x^2 - 4 + x + 3 = x^2 + x - 1$ with the domain $\mathbb{R}$

$(f-g)(x) = f(x) - g(x) = x^2 - 4 - (x+3) = x^2 - x - 7$ with th e domain $\mathbb{R}$,

$(fg)(x) = f(x)g(x) = (x^2 - 4)(x+3) = x^3 + 3x^2 - 4x - 12$ with the domain $\mathbb{R}$

$(f/g)(x) = \dfrac{f(x)}{g(x)} = \dfrac{x^2 - 4}{x+3}$ with domain $(-\infty, -3) \cup (-3, \infty)$ since -3 makes t he denominator of the new function equal to zero.

b. $(f+g)(1) = f(1) + g(1) = 1^2 - 4 + 1 + 3 = 1$

$(f-g)(1) = f(1) - g(1) = 1^2 - 4 - (1+3) = -7$

$(fg)(1) = f(1)g(1) = (1^2 - 4)(1+3) = -12$

$(f/g)(1) = f(1)/g(1) = (1^2 - 4)/(1+3) = -3/4$

Example 54 Given that $f(x) = 2$, $g(x) = 3x^2 - 11$, $h(x) = \sqrt{2x-6}$ and $m(x) = \dfrac{f(x)+g(x)}{h(x)} + f(x)g(x)$, find $m(5)$ and the domain of m.

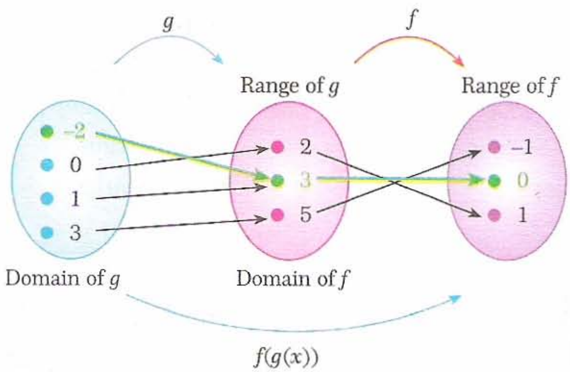
Solution $m(x) = \dfrac{f(x)+g(x)}{h(x)} + f(x)g(x) = \dfrac{2+3x^2-11}{\sqrt{2x-6}} + 2(3x^2-11) = \dfrac{3x^2-9}{\sqrt{2x-6}} + 6x^2 - 22.$

$$m(5) = \frac{3\cdot 5^2 - 11}{\sqrt{2\cdot 5-6}} + 6\cdot 5^2 - 22 = 160.$$

The domain of function m can directly be found by considering the final formula for $m(x)$. The only condition that we should deal with is $2x - 6 > 0$. So, $D(m) = (3, \infty)$.

B. COMPOSITION OF FUNCTIONS

Aside from the four basic operations there is a very important way of combining functions to get a new function. We call this a **composition**. This operation is illustrated as follows:

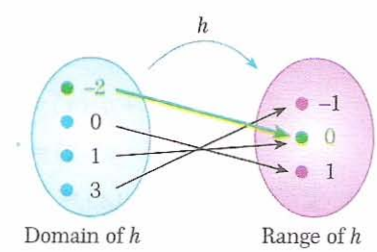


Here we see two different functions f and g. Their common point is that the range of one is the domain of the other. Because of that fact the domain of the function g is linked with the range of the function f. Consider $x = -2$:

$$g(-2) = 3$$
$$f(g(-2)) = f(3) = 0$$

In other words, -2 finally maps to 0. Note that to find the final mapping we first use g and then f. If we name that final mapping as h, then $h(-2) = 0$ where $h(x) = f(g(x))$.

That final mapping h is another function and is the composition of f and g as shown below on the right:

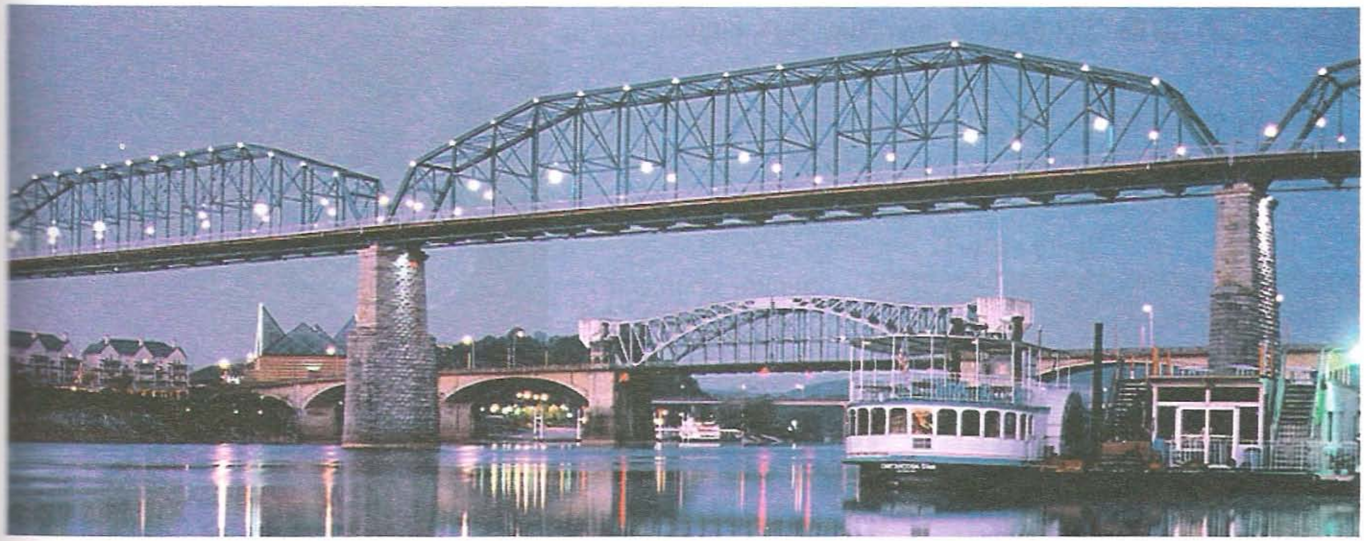


> $f(g(x))$ means g is applied first, f is applied second.

> Read $f(g(x))$ as "f of g of x".

As an algebraic example if $f(x) = \sqrt{x}$ and $g(x) = 3x + 1$, we know that $f(x)$ is the rule "take the square root of the number x" and $g(x)$ is the rule "multiply the number x by 3 and add 1". In that case $f(g(x))$ means you must "first multiply the number x by 3 and add 1, then take the square root of the result" so we can define $h(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{3x+1}$. Here we say that $h(x)$ is a composition of $f(x)$ and $g(x)$, that is $h(x)$ is a composite function.

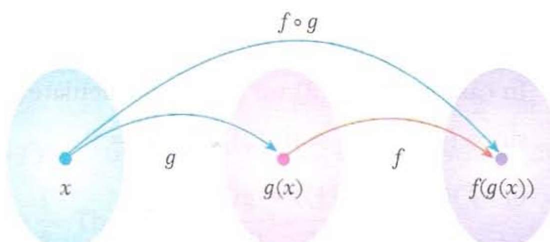
The domain of $f(g(x))$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.



Composite functions are composed of "bridges" that link the domain of "first" with the range of "last".

Definition composition

$f(g(x))$ is defined as **composition** of f and g. It is also denoted by $(f \circ g)(x)$ or simply $f \circ g$.



Arrow diagram for $(f \circ g)(x)$

Example 55 Given that $f(x) = x^2 + 3x$, $g(x) = x - 4$, find $f(g(x))$ and $g(f(x))$.

Solution 1 $f(g(x)) = g(x)^2 + 3g(x) = (x - 4)^2 + 3(x - 4) = x^2 - 8x + 16 + 3x - 12 = x^2 - 5x + 4$

$g(f(x)) = f(x) - 4 = (x^2 + 3x) - 4 = x^2 + 3x - 4$

So $f(g(x)) = x^2 - 5x + 4$ and $g(f(x)) = x^2 + 3x - 4$.

What can you say about $f(g(x))$ and $g(f(x))$? Is it true that they are the same?

Solution 2 $f(g(x)) = f(x - 4) = (x - 4)^2 + 3(x - 4) = x^2 - 5x + 4$

$g(f(x)) = g(x^2 + 3x) = (x^2 + 3x) - 4 = x^2 + 3x - 4$

Note

In general, $f(g(x)) \neq g(f(x))$.

Example 56 Let $f(x) = x + 1$, $g(x) = x^3$. Find:

a. $(f \circ g)(2)$

b. $(g \circ f)(1)$

c. $(f \circ f)(5)$

d. $(g \circ g)(-1)$

Composite functions are formed of functions that are inside another function.

Solution 1

a. $(f \circ g)(x) = f(g(x)) = g(x) + 1 = x^3 + 1$

$(f \circ g)(2) = 2^3 + 1 = 9$

b. $(g \circ f)(x) = g(f(x)) = (f(x))^3 = (x + 1)^3$

$(g \circ f)(1) = (1 + 1)^3 = 8$

c. $(f \circ f)(x) = f(f(x)) = f(x) + 1 = x + 2$

$(f \circ f)(5) = 5 + 2 = 7$

d. $(g \circ g)(x) = g(g(x)) = (g(x))^3 = x^9$

$(g \circ g)(-1) = (-1)^9 = -1$

> $f \circ g$ means g is applied first, f is applied second and in general $f \circ g \neq g \circ f$.

Solution 2 Note that these values can be calculated without finding the formula.

a. In case of $(f \circ g)(2)$ we may first calculate $g(2) = 8$ and then $f(8) = 9$ to find the answer since $(f \circ g)(2) = f(g(2)) = f(8) = 9$.

b. $(g \circ f)(1) = g(f(1)) = g(2) = 8$

c. $(f \circ f)(5) = f(f(5)) = f(6) = 7$

d. $(g \circ g)(-1) = g(g(-1)) = g(-1) = -1$

Example 57 Let $f(x) = \dfrac{x}{x-1}$, $g(x) = x^5$ and $h(x) = x^2 + x$. Find $f \circ g \circ h$.

Solution 1 Here we have a composition of three functions.

$$(f \circ g \circ h)(x) = f(g(h(x))) = \frac{g(h(x))}{g(h(x))-1} = \frac{(h(x))^5}{(h(x))^5-1} = \frac{(x^2+x)^5}{(x^2+x)^5-1}.$$

Solution 2 $$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2 + x)) = f((x^2 + x)^5) = \frac{(x^2+x)^5}{(x^2+x)^5-1}.$$

Example 58 Write the function $h(x) = \sqrt{5x+3}$ as a composition of two functions.

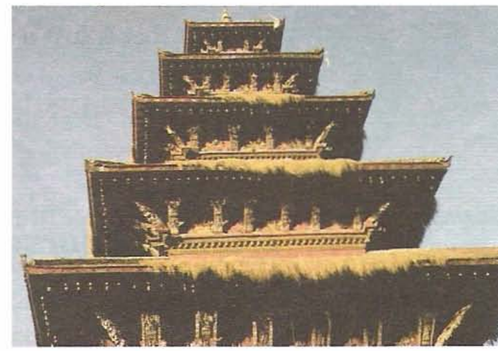
Solution Let us read what the formula tells us to do: "first add 3 to five times the number x and then take the square root of all". Let $f(x) = 5x + 3$ and $g(x) = \sqrt{x}$. Note that f is applied first, g is applied second. So $h(x) = (g \circ f)(x)$.

The composite function form is never unique.

For example, consider the previous example:

If n is any nonzero integer, we could choose $f(x) = (5x + 3)^{1/n}$ and $g(x) = x^{n/2}$.

Thus, there are an infinite amount of composite function forms. Generally, our aim is to choose a formula such that the expression for the function is simple, as in the above example.



Example 59 Given $f(x) = 3x + 1$ and $g(x) = 2x - m$ such that $(f \circ g)(x) = (g \circ f)(x)$, find $g(\frac{1}{10})$.

Solution $(f \circ g)(x) = f(g(x)) = 3 \cdot g(x) + 1 = 3(2x - m) + 1 = 6x - 3m + 1$

$(g \circ f)(x) = g(f(x)) = 2 \cdot f(x) - m = 2(3x + 1) - m = 6x + 2 - m$

Since $(f \circ g)(x) = (g \circ f)(x)$, $6x - 3m + 1 = 6x + 2 - m$ which gives $m = -\frac{1}{2}$.

So we have $g(x) = 2x + \frac{1}{2}$. Therefore, $g(\frac{1}{10}) = \frac{7}{10}$.

Example 60 If the rule for function $f \circ g \circ h$ is "first take the square root of a number plus twenty-five, second divide the new number plus 2 by itself, third take the cube of the newest number" and $p(x) = 3x^2 + 6x$. Find $(f \circ g \circ h)(x)$ and $(h \circ p \circ f)(-2)$.

Solution First of all let us find formulae for f, g and h.

Since $f \circ g \circ h$ means h is applied first, g second and f third we have

$h(x) = \sqrt{x+25}$, $g(x) = \dfrac{x+2}{x}$, $f(x) = x^3$.

According to the rule, $(f \circ g \circ h)(x) = \left(\dfrac{\sqrt{x+25}+2}{\sqrt{x+25}}\right)^3$.

Now let us find $(h \circ p \circ f)(-2)$:

$(h \circ p \circ f)(-2) = h(p(f(-2))) = h(p(-8)) = h(144) = 13.$

Example 61 Given that $f(g(x)) = 4x - 1$ and $g(x) = x + 2$, find $f(x)$.

Solution Since $f(g(x)) = 4x - 1$ and $g(x) = x + 2$ we have $f(x + 2) = 4x - 1$.

Let $x + 2 = a$. Then $x = a - 2$.

Substituting $x = a - 2$ in $f(x + 2) = 4x - 1$ we get

$f(a - 2 + 2) = 4(a - 2) - 1$ or $f(a) = 4a - 9$.

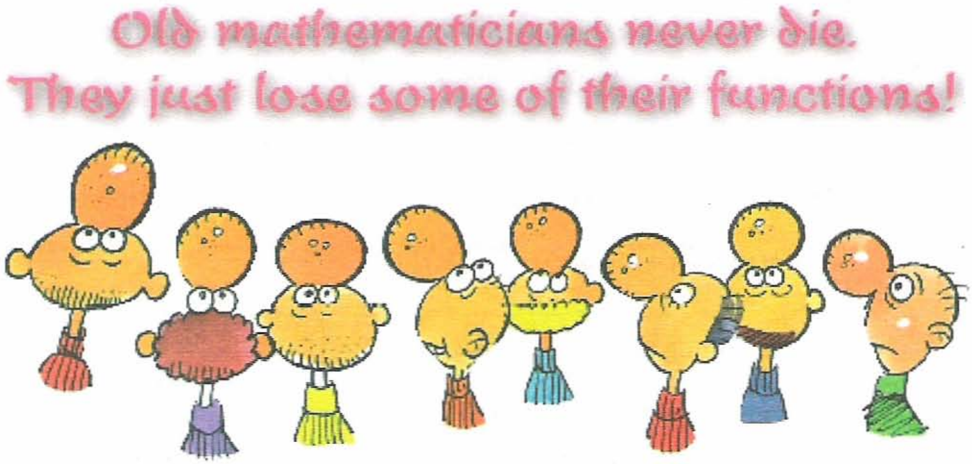
Since this is just a matter of notation in the final formula instead of a we choose x to get $f(x) = 4x - 9$.

Check Yourself 13

1. Given that $f(x) = \dfrac{x+1}{2x-4}$ and $g(x) = x$, find $f + g, f - g, fg, f / g$.
2. Given that $f(x) = x^2$ and $g(x) = 3x + 4$, find $(f \circ g)(1)$ and $(g \circ f)(1)$.
3. Write the function $h(x) = (2x - 7)^5$ as a composition of two functions such that $h(x) = (f \circ g)(x)$.

Answers

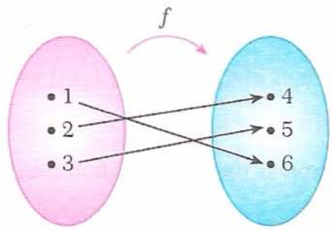
1. $\dfrac{2x^2 - 3x + 1}{2x - 4}$, $\dfrac{-2x^2 + 5x + 1}{2x - 4}$, $\dfrac{x^2 + x}{2x - 4}$, $\dfrac{x + 1}{2x^2 - 4x}$ 2. 49, 7 3. $f(x) = x^5$, $g(x) = 2x - 7$

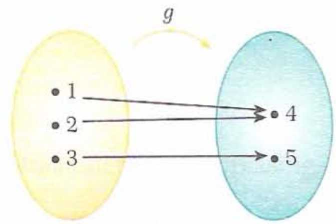


C. INVERSE OF A FUNCTION

1. One-to-one Functions

Consider the two functions represented by a map as shown below:





Each number in the domain of f takes a different value from the range. But it is not the same for the function g since 1 and 2 take the same value in the range. We name a function one-to-one if each element in the domain corresponds to exactly one element from the range. So here f is **one-to-one** and g is not one-to-one. To understand whether a given formula belongs to a one-to-one function or not we need an algebraic tool. The following definition for one-to-one function meets our needs:

Definition one-to-one function

A function f is **one-to-one** if, for each $x_1 \neq x_2$ in its domain, $f(x_1) \neq f(x_2)$.

Example 62 Which one of the following is a one-to-one function?

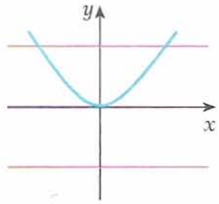
a. $f(x) = 2x - 1$ b. $f(x) = x^2$

Solution Let us use the definition for one-to-one function to find out whether they support the definition or not:

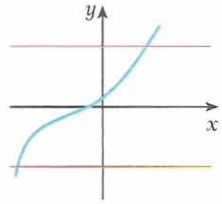
a. Let $x_1 \neq x_2$. Then $f(x_1) = 2x_1 - 1$ and $f(x_2) = 2x_2 - 1$. Let's assume that $f(x_1) = f(x_2)$, so $2x_1 - 1 = 2x_2 - 1$ which means $x_1 = x_2$. But this is wrong since we let $x_1 \neq x_2$! So our assumption that $f(x_1) = f(x_2)$ is also wrong. That means when $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$. Therefore, the function is one-to-one.

b. Clearly, if we choose $-2 \neq 2$ we get $f(-2) = f(2) = 4$. Although we can find an infinite amount of such examples, just one of them is enough to decide that function is not one-to-one.

We know that the graph of a relation is a function if any vertical line crosses the curve at most once. Similarly, a function is **one-to-one** if any horizontal line crosses the curve at most once. For example:



Graph of
a function which is not one-to-one



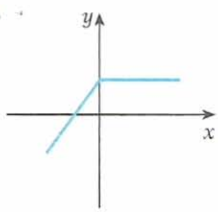
Graph of
a one-to-one function

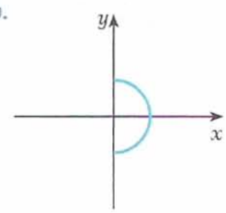
HORIZONTAL LINE TEST FOR THE ONE-TO-ONE FUNCTIONS

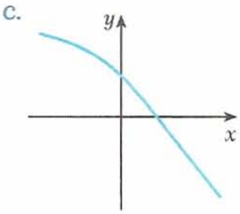
A function is one-to-one if, and only if, no horizontal line crosses its graph at more than one point.

Note that in the previous example we could have found the answers without using the definition but just by plotting their graphs and applying the horizontal line test.

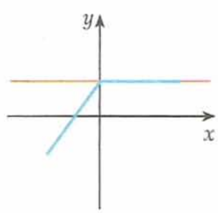
Example 63 Which one of the following graphs belong to one-to-one functions?

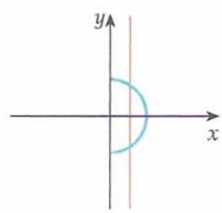
a. 

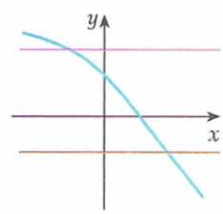
b. 

c. 

Solution







a. This graph does not belong to a one-to-one function.

b. This graph does not belong to a function, so we can not talk about one-to-one function.

c. This graph belongs to a one-to-one function.

Note

A function that is only increasing or decreasing in its domain is one-to-one.

Check Yourself 14

1. Prove that $f(x)=\dfrac{3x-1}{5}$ is a one-to-one function.
2. Draw a graph which belongs to a one-to-one function.
3. Draw a graph which does not belong to a one-to-one function.

Answers

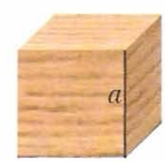
1. Compare $f(x_1)$ and $f(x_2)$ when $x_1 \neq x_2$.
2. Consider a function which is either increasing or decreasing.
3. Consider a function which is sometimes increasing and sometimes decreasing.

2. Definition for The Inverse of a Function

As mentioned earlier in this book, many mathematical relations can be modelled as functions. For example,

$C = f(r) = 2\pi r$

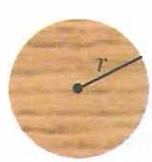
The circumference of a circle is a function of the radius r.



$V = g(a) = a^3$

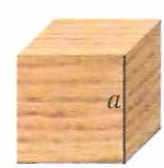
The volume of a cube is a function of the edge a.

In many cases we are interested in reversing this correspondence determined by a function:



$r = m(C) = \dfrac{C}{2\pi}$

The radius of a circle is a function of the circumference C.

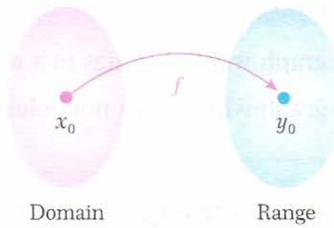


$a = n(V) = \sqrt[3]{V}$

The edge of a cube is a function of the volume V.

As illustrated above, reversing the relation between two quantities produces a new function.

Recall that a function is a relation that assigns to each element in the domain exactly one element from the range.



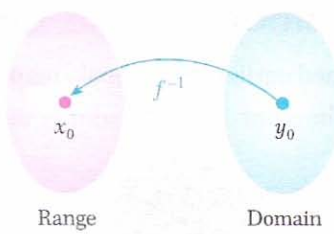
In the figure above we see an arrow diagram for function f. So "Using the rule f we say that x_0 becomes y_0". We symbolize this as "$f(x_0) = y_0$".

Read $f^{-1}(x)$ as "f inverse of x".

$f^{-1}(x)$ doesn't mean $\frac{1}{f(x)}$.

The reciprocal $\frac{1}{f(x)}$ is written as $(f(x))^{-1}$.

Now consider this question: Using the rule f what gives us y_0? Of course the answer is x_0. So "Using the rule f to get y_0 we need x_0". We need a new notation so that "x_0" will be our answer. And we symbolize this fact as "$f^{-1}(y_0) = x_0$".



Note

The domain of a function is the same as the range of its inverse.

The range of a function is the same as the domain of its inverse.

The domain and the range change their places in an inverse of a function just as the server and the opponent change their places in a table tennis game.

Example 64 Given the function $f = \{(0, 2), (-1, 4), (4, 6), (5, 5)\}$,

a. find f^{-1}. b. find $f^{-1}(4)$.

Solution a. Interchange the x and y-coordinates of each ordered pair of f to find f^{-1}:

$f^{-1} = \{(2, 0), (4, -1), (6, 4), (5, 5)\}$

b. To find the value of $f^{-1}(4)$, notice that $f^{-1}(4)$ is the second coordinate when the first coordinate is 4 in the function f^{-1}. So $f^{-1}(4) = -1$.

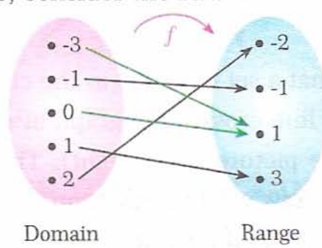
Example 65 If $f(x) = 2x - 3$, find $f^{-1}(5)$.

Solution If we substitute 5 in place of x we find $f(5) = 2 \cdot 5 - 3 = 7$ but not $f^{-1}(5)$!

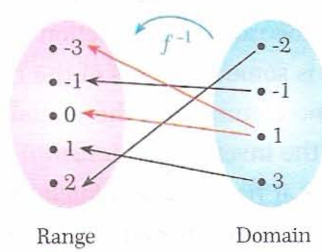
Finding $f^{-1}(5)$ means finding the x value for which $f(x) = 2x - 3$ gives 5 as a result.

$2x - 3 = 5$, so $x = 4$ or $f^{-1}(5) = 4$.

If f^{-1} also supports the definition for a function, we will call f^{-1} as the **inverse** of f. But this is not always possible. For example, consider the relation below:



Clearly, this is a function since each element is assigned to exactly one element. Now let's consider its inverse:



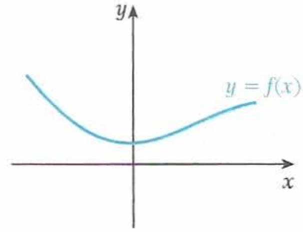
As we see $f^{-1}(1) = -3$ or $f^{-1}(1) = 0$ is not possible for a function (*just one element must be assigned for each element in the domain*). So although f is a function, f^{-1} is not a function. That means the inverse of f doesn't exist.

Note

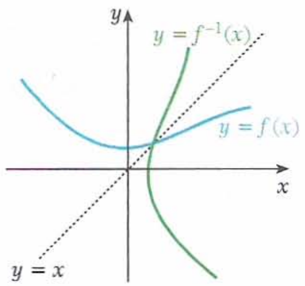
It is not true that every function has an inverse.

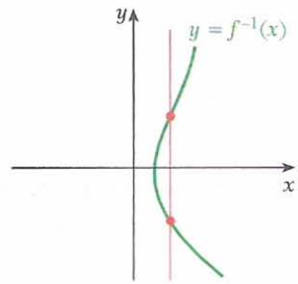
For the inverse of a function to be defined, it is necessary that different elements in the domain always give different values. As we know such functions are called one-to-one functions.

As another example let us consider the following function graph of which is given below:



To find the inverse, we use the same procedures that we used for relations. Drawing the reflection with respect to $y = x$ we get the following picture (below left):





We know that a set of points in the coordinate plane is the graph of a function if, and only if, no vertical line crosses the graph at more than one point. But that rule doesn't hold for f^{-1} (look at the picture above right). That means the inverse of f doesn't exist. Note that the inverse we draw will be a function if f has no horizontal line crossing the graph at more than one point, that is if f is one-to-one.

We know that the range of a function becomes the domain of its inverse. We also know that no element from the domain of a function must be left unassigned.

Sometimes the range of a function is explicitly so that it is larger than the function's its real range, that is some elements of the range are not used. In that case when we talk about the inverse some elements in the domain of inverse will be unassigned. This will result in an absence of the inverse function.

For example, if $f(x) = x + 2$ such that $f: \mathbb{Z} \to \mathbb{R}$, although the range seems to be $\mathbb{R}$ it is in fact $\mathbb{Z}$ (Note that when we put any integer in $x + 2$, we always get other integers).

To guarantee that we will not face such functions we must be sure that any element in the range is assigned by an element from the domain. Such functions are called **onto** functions.

CRITERIA FOR EXISTENCE OF INVERSE OF A FUNCTION

A function has an inverse if, and only if, it is one-to-one and onto.

If a function is given by only formula or graph (where no explicit range is given), it is naturally an onto function. So there is no need to think about this detail to define its inverse.

Now we can give the formal definition for the inverse of a function:

Definition

inverse function

Let f be a one-to-one and onto function with the domain A and the range B. Then its **inverse function** f^{-1} has the domain B and the range A such that $f(x) = y \Leftrightarrow f^{-1}(y) = x$.

By definition the inverse function f^{-1} undoes what f does. That is, if we take x, apply f, and then apply f^{-1}, we arrive back at x where we started. Similarly, f undoes what f^{-1} does. That is why f and f^{-1} are the inverses of each other.

Note

$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Example 66 Prove that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of each other.

Solution When we find $f(g(x)) = (\sqrt[3]{x})^3 = x$, we can see that $f(x)$ and $g(x)$ are the inverses of each other. Note also that $g(f(x)) = x$.

Example 67 If $f(x) = x + 2$ and $f^{-1}(x) = x - a$, find a.

Solution We know that $f(f^{-1}(x)) = x$. So $f(x - a) = x$ or $(x - a) + 2 = x$. That gives $a = 2$.

3. Finding The Inverse of a Function

Given the graph of a function, to find the inverse we use the following procedure:

1. Verify that the function is one-to-one by applying the horizontal line test.
2. Take symmetry of the graph of the function with respect to the line $y = x$.

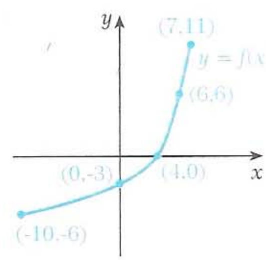
Example 68 Answer the following using the graph on the right:

a. Does f have an inverse? Why?

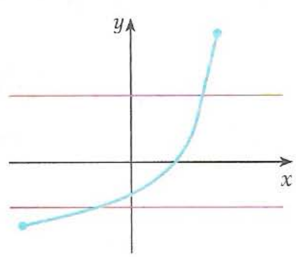
b. Draw the graph of the inverse function f^{-1}.

c. Find $f^{-1}(-6)$, $f^{-1}(-3)$, $f^{-1}(0)$, $f^{-1}(6)$, $f^{-1}(11)$.

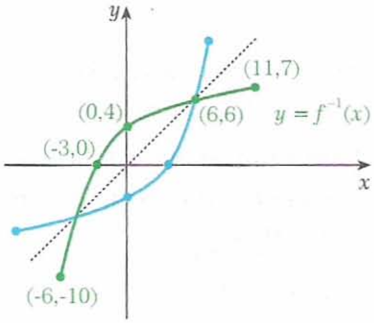
d. Find the domain and the range of f^{-1}.

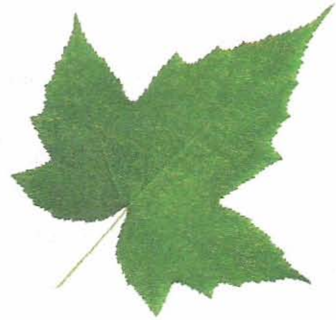


Solution a. Applying the horizontal line test we can see that f is one-to-one. So it has an inverse



b. We draw the line $y = x$ and reflect f with respect to it to draw the graph of f^{-1}:





c. $f^{-1}(6) = -10,\ f^{-1}(-3) = 0,\ f^{-1}(0) = 4,\ f^{-1}(6) = 6,\ f^{-1}(11) = 7.$

d. f^{-1} can take any x between –6 and 11, inclusive, so the domain is $[-6, 11]$. The value y of f^{-1} can be any number x between –10 and 7, inclusive, so the range is $[-10, 7]$.

Given the formula for a function, to find the inverse we use the following procedure:

1. Verify that the function is one-to-one.
2. Solve the equation $y = f(x)$ for x and interchange x and y in the end.

Example 69 Given that $f(x) = 2x + 5$, answer the following:

a. Does f have an inverse? Why?

b. Find the formula of the inverse function f^{-1}.

c. Find $f^{-1}(19), f^{-1}(1)$.

d. Find the domain and the range of f^{-1}.

Solution a. f is a linear function which is always increasing. That means it is one-to-one and so it has an inverse.

b. Let $y = 2x + 5$. Then $x = \dfrac{y-5}{2}$.

Interchanging x and y we have the inverse function as $y = \dfrac{x-5}{2}$ or $f^{-1}(x) = \dfrac{x-5}{2}$.

c. Using the formula $f^{-1}(x) = \dfrac{x-5}{2}$ we get $f^{-1}(19) = \dfrac{19-5}{2} = 7$ and $f^{-1}(1) = \dfrac{1-5}{2} = -2$. Note that we could also solve equations $2x + 5 = 19$ and $2x + 5 = 1$ to find $f^{-1}(19)$ and $f^{-1}(1)$, respectively.

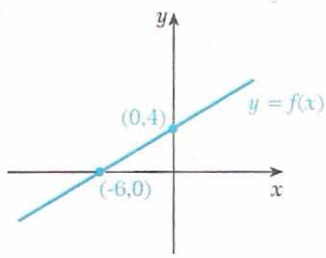
d. Clearly, $D(f^{-1}) = \mathbb{R}$ and $E(f^{-1}) = \mathbb{R}$.

FINDING THE INVERSE OF A FUNCTION

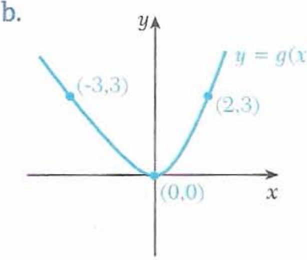
- To plot the graph of the inverse of a function, reflect the graph of the function with respect to the line $y = x$.
- To find the formula for the inverse of a function, solve the equation $y = f(x)$ for x and interchange x and y.

Example 70 If possible draw the graphs of inverses for the following functions:

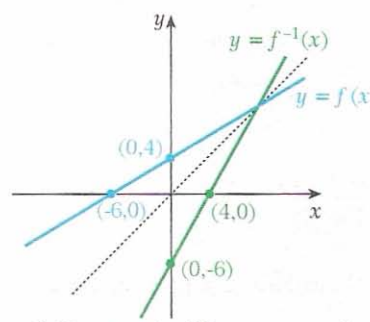
a.



b.



Solution a. Clearly (*horizontal line test*), f is one-to-one. So the inverse is the symmetry of f with respect to the line $y = x$:



b. g is not one-to-one (*horizontal line test*). We cannot draw the graph of the inverse function since it doesn't exist.

Example 71 Given that the following one-to-one functions, find their inverses.

a. $f(x) = 4 - 3x$ b. $f(x) = \dfrac{2x-6}{3}$ c. $g(x) = x^3 - 5$

Solution Since we know that the functions are one-to-one, we can find their inverses directly

a. Let $y = 4 - 3x$. Then $x = \dfrac{4-y}{3}$. Interchanging x and y: $y = \dfrac{4-x}{3}$ or $f^{-1}(x) = \dfrac{4-x}{3}$.

b. Let $y = \dfrac{2x-6}{3}$. Then $x = \dfrac{3y+6}{2}$. Interchanging x and y: $y = \dfrac{3x+6}{2}$ or $f^{-1}(x) = \dfrac{3x+6}{2}$.

c. Let $y = x^3 - 5$. Then $x = \sqrt[3]{y+5}$. Interchanging x and y: $y = \sqrt[3]{x+5}$ or $g^{-1}(x) = \sqrt[3]{x+5}$.

Example 72 If possible find the inverses of the following functions:

a. $f(x)=\dfrac{2x+4}{3x-5}$ b. $f(x)=\sqrt[3]{2x-1}+7$ c. $f(x) = x^2 - 2x$

Solution a. We can verify that $f(x)=\dfrac{2x+4}{3x-5}$ is one-to-one as follows:

Let $x_1 \neq x_2$. Then assume that $f(x_1) = f(x_2)$. So,

$$\frac{2x_1+4}{3x_1-5}=\frac{2x_2+4}{3x_2-5}$$
$$6x_1x_2-10x_1+12x_2-20=6x_1x_2+12x_1-10x_2-20$$
$$22x_2=22x_1$$
$$x_2=x_1.$$

But this is not correct since we let $x_1 \neq x_2$. This means that if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. So f is one-to-one. Now let's find its inverse:

$$f(x)=y=\frac{2x+4}{3x-5}$$
$$3xy-5y=2x+4$$
$$3xy-2x=4+5y$$
$$x(3y-2)=4+5y$$
$$x=\frac{4+5y}{3y-2}$$

Interchanging x and y we have $y=\dfrac{4+5x}{3x-2}$ or $f^{-1}(x)=\dfrac{4+5x}{3x-2}$.

b. We leave the verification of the fact that f is one-to-one to student and proceed to finding its inverse.

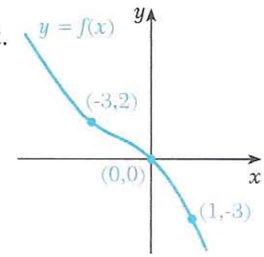
$$y=\sqrt[3]{2x-1}+7$$
$$y-7=\sqrt[3]{2x-1}$$
$$(y-7)^3=2x-1$$
$$\frac{(y-7)^3+1}{2}=x$$

Interchanging x and y we have $y=\dfrac{(x-7)^3+1}{2}$ or $f^{-1}(x)=\dfrac{(x-7)^3+1}{2}$.

c. Choosing $x = 0$ and $x = 2$ we can realize that f will give the same value. So f is not one-to-one. Proof of this using definition is left to the student in exercises. As a result f does not have an inverse.

Check Yourself 15

1. If $f(2) = 3, f(5) = 6, f(6) = -1$, find $f^{-1}(-1), f^{-1}(3), f^{-1}(6)$.
2. If possible draw the inverse of f graph of which is given on the right.

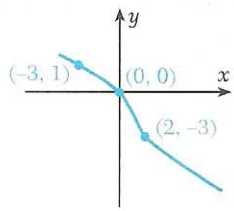


3. Find the inverse of $f(x) = \dfrac{2x+3}{7}$.

Answers

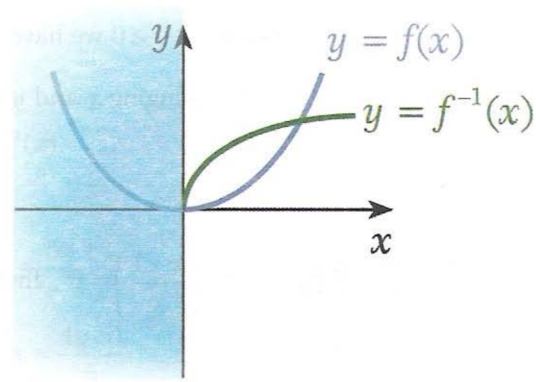
1. 6, 2, 5

2.



3. $f^{-1}(x) = \dfrac{7x-3}{2}$

The functions which are not one-to-one can have their domains restricted so that they become one-to-one. As a result their inverses will then be functions.



Example 73 Given that $f(x) = x^2 + 1$,

a. find the largest possible domain so that the inverse exists.

b. find $f^{-1}(x)$ using the new domain.

Solution a. The inverse exists if we have a one-to-one function. Note that for each value of x and its negative we have the same value of y. So, if we ignore the negative x values, we will have a one-to-one function. That means to have an inverse, domain must be restricted to $[0, \infty)$.

b. Let $y = x^2 + 1$ and we will try to solve the equation for x: $x^2 = y - 1$, so $x = \mp\sqrt{y-1}$.

If we interchange x and y we have $y = \mp\sqrt{x-1}$. Note that the range of the inverse function is the domain of the original function, that is $[0, \infty)$. So $y = \sqrt{x-1}$. Therefore, the inverse function is $f^{-1}(x) = \sqrt{x-1}$.

Example 74 Prove that $f(x) = x^2 - 2x - 2$ has an inverse if $D(f) = [1, \infty)$ and then find its inverse.

Solution $f(x) = x^2 - 2x - 2 = x^2 - 2x + 1 - 3 = (x-1)^2 - 3.$

Here $(x-1)^2$ is equal to the same number for $x - 1 = x_0$ and $x - 1 = -x_0$ for any $x_0 > 0$.

This fact prevents $f(x) = (x-1)^2 - 3$ from being a one-to-one function. But if we guarantee that $x - 1$ is never negative, then the function will be one-to-one. And that is possible when $x - 1 \geq 0,\ x \in [1, \infty)$.

To find the inverse, we solve $y = (x-1)^2 - 3$ for x:

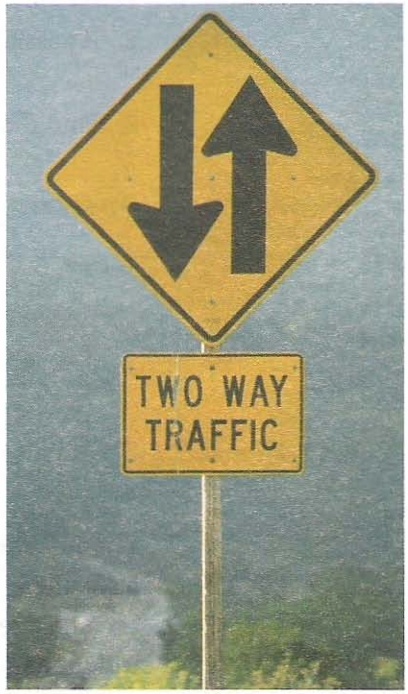
$$y = (x-1)^2 - 3$$

$$y + 3 = (x-1)^2$$

$$\sqrt{y+3} = |x-1|$$

Since $x - 1 \geq 0$ we have $\sqrt{y+3} = x - 1$ or $\sqrt{y+3} + 1 = x$

Interchanging x and y we have $f^{-1}(x) = \sqrt{x+3} + 1$.



Example 75 If $f^{-1}\left(\dfrac{x+1}{x}\right) = x^3$, find $f^{-1}(2) + f(8)$.

Solution Let us find $f^{-1}(2)$. We don't have the formula for $f^{-1}(x)$ but for $f^{-1}(\frac{x+1}{x})$. The expression inside the brackets must be equal to 2 since we are looking for $f^{-1}(2)$.

When we solve $\dfrac{x+1}{x} = 2$, we find $x = 1$.

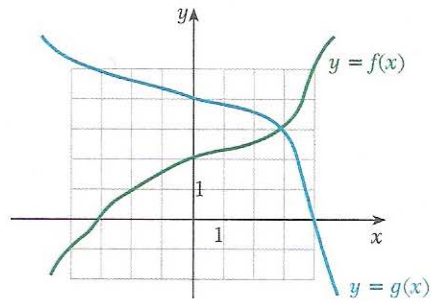
That means when $x = 1$, $f^{-1}(\frac{x+1}{x}) = f^{-1}(2) = 1^3 = 1$.

Finding $f(8)$ is simple. Note that f is inverse of f^{-1}. So if we need $f(8)$, the result of $f^{-1}(\frac{x+1}{x}) = x^3$ must be equal to 8, that is $f^{-1}(\frac{x+1}{x}) = 8$, where $\dfrac{x+1}{x}$ will be $f(8)$ for the corresponding x value.

So, $x^3 = 8$, $x = 2$ and $\dfrac{x+1}{x} = \dfrac{2+1}{2} = \dfrac{3}{2}$.

Finally, $f^{-1}(2) + f(8) = 1 + \dfrac{3}{2} = \dfrac{5}{2}$.

Example 76 Using the graph below find $(f \circ g^{-1} \circ f^{-1})(5) + (f \circ g \circ f)(-3)$.



Solution Note that we don't have the graph of any inverse function and we don't need them at all. To find the value of a function just find the y-value for the given x-value on the graph. To find the value of an inverse function just find the x-value for the given y-value on the graph.

$(f \circ g^{-1} \circ f^{-1})(5) = f(g^{-1}(f^{-1}(5))) = f(g^{-1}(4)) = f(0) = 2$

$(f \circ g \circ f)(-3) = f(g(f(-3))) = f(g(0)) = f(4) = 5$

So $(f \circ g^{-1} \circ f^{-1})(5) + (f \circ g \circ f)(-3) = 2 + 5 = 7$.

Example 77 If $(f \circ g)(x) = 5x - 2$ and $f(x) = 4x - 3$, find $g^{-1}(x)$.

Solution $(f \circ g)(x) = f(g(x)) = 4 \cdot g(x) - 3 = 5x - 2$

Solving the last equation for $g(x)$ we have $g(x) = \dfrac{5x+1}{4}$

To find the inverse, we solve $y = \dfrac{5x+1}{4}$ for x: $x = \dfrac{4y-1}{5}$

Interchanging x and y we have $g^{-1}(x) = \dfrac{4x-1}{5}$.

Example 78 Given $f(x) = 2x - 7$, find $f^{-1}(4x + 1)$.

Solution Let us find $f^{-1}(x)$: $f(x) = 2x - 7 = y$

$$x = \frac{y+7}{2}$$

$$f^{-1}(x) = \frac{x+7}{2}$$

Now let us find $f^{-1}(4x + 1)$: $f^{-1}(4x+1) = \dfrac{(4x+1)+7}{2} = 2x + 4$

✓

Example 79 Given that $f(x) = 3x + 1$ and $g(x) = 2x + 2$, find

a. $(f \circ g)^{-1}$ b. $g^{-1} \circ f^{-1}$

Solution a. Let us find $f \circ g$.

$y = (f \circ g)(x) = f(g(x)) = 3(2x + 2) + 1 = 6x + 7$

To find the inverse, $x = \dfrac{y-7}{6}$

$$(f \circ g)^{-1}(x) = \frac{x-7}{6}.$$

b. Let us find f^{-1} and g^{-1}.

$y = f(x) = 3x + 1$

To find the inverse, $x = \dfrac{y-1}{3}$

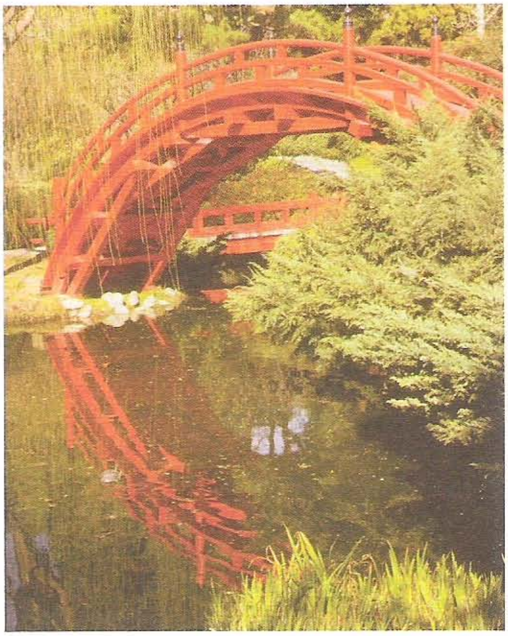
$$f^{-1}(x) = \frac{x-1}{3}.$$

$y = g(x) = 2x + 2.$

To find the inverse, $x = \dfrac{y-2}{2}$

$$g^{-1}(x) = \frac{x-2}{2}.$$

$$(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x)) = \frac{\frac{x-1}{3} - 2}{2} = \frac{x-7}{6}$$



Note that we have $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

Is it always correct? Why?

Check Yourself 16

1. Given that $f(x) = x^2 - 4$, restrict the domain so that the inverse exists and find its formula.
2. Given that $f(2x + 1) = x - 5$, find $f^{-1}(0)$.
3. Given that $f(g(x)) = 2x + 1$ and $g(x) = x + 4$, find $f^{-1}(x)$.
4. Given that $f(x) = \dfrac{x-1}{4}$ and $g(x) = 3x$, find $(f \circ g)^{-1}$ and $g^{-1} \circ f^{-1}$.

Answers

1. $[0, \infty)$, $\sqrt{x+4}$ 2. 11 3. $\dfrac{x+7}{2}$ 4. $\dfrac{4x+1}{3}$, $\dfrac{4x+1}{3}$

EXERCISES 3

A. Basic Operations

1. Find $f+g, f-g, fg, f/g$ for the following functions.

 a. $f(x) = x + 1,\ g(x) = x^2 - 1$

 b. $f(x) = x^3 + 3x^2,\ g(x) = x^2 + 5x$

 c. $f(x) = x+3,\ g(x) = \sqrt{x+2}$

 d. $f(x) = \sqrt{x-1},\ g(x) = \sqrt{x+1}$

2. If $f(x) = 3x+4,\ g(x) = x^2+x,\ h(x) = \dfrac{1}{x},$ find the following functions.

 a. $(f+g)(3)$ b. $\left(\dfrac{f-h}{g}\right)(-1)$

 c. $(hg-f)(4)$

B. Composition of Functions

3. Find $f \circ f,\ f \circ g,\ g \circ f,\ g \circ g$ for the following functions.

 a. $f(x) = \dfrac{x+1}{x-1},\ g(x) = x^2 + x$

 b. $f(x) = x^2 - 2x + 1,\ g(x) = x + 1$

 c. $f(x) = \dfrac{2x+3}{x},\ g(x) = \dfrac{x+3}{3x-2}$

4. Express $h(x)$ in terms of $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$.

 a. $h(x) = \sqrt{2x-4}$

 b. $h(x) = \left(\dfrac{x-4}{5}\right)^7$

 c. $h(x) = (2x^2 - x)^3 - (2x^2 - x) + 4$

5. Find the required values using the given data:

 a. $f(x) = \dfrac{3x+5}{x+1}$ and $g(x) = \dfrac{x+2}{x-1}$,
 $(f \circ g \circ f)(0) = ?$

 b. $f(2x+1) = 3x - 1$ and $g\left(\dfrac{x+1}{x-1}\right) = x^2 + 1$,
 $(f \circ g)(-1) = ?$

 c. $f(3x-1) = \dfrac{x+3}{2x-1}$ and $g(x+1) = \dfrac{x-2}{x-1}$,
 $(f \circ g)(1) = ?$

 d. $f(x) = \begin{cases} 2x+1 & \text{if } x > 2 \\ x-2 & \text{if } x \le 2 \end{cases}$ and

 $g(x) = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \le 0 \end{cases}$
 $(f \circ g \circ f)(-2) = ?$

6. If $f(x) = x^3 - 3x^2 + 3x - 1$ and $g(x) = \dfrac{2x+1}{x+3}$, solve $(f \circ g)(x) = 0$.

7. If $(f \circ g)(x) = 5g(x) + 4$ and
 $(g \circ f)(x) = 3f(x) - 1$,
 find a such that $(f+g)(a) = 19$.

8. If $f(x) = \dfrac{1-x}{1+x}$, find $\left(\underbrace{f \circ f \circ \ldots \circ f}_{2004 \text{ times}}\right)(x)$.

C. Inverse of a Function

9. State whether the following relations are one-to-one functions or not.

 a. $\{(x, 1), (y, 1), (z, 2), (k, 3)\}$

 b. $\{(1, 1), (2, 2), (3, 3), (4, 1)\}$

 c. $\{(a, b), (b, a), (c, d), (d, c)\}$

10. State whether the following functions are one-to-one or not.

a. $f(x) = x^2 - 1,\ D(f) = \{0, 1, 2, 3\}$

b. $f(x) = x^2,\ D(f) = \mathbb{R}^+$

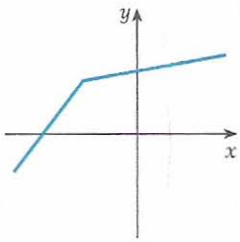
c. $f(x) = x^2$

d. $f(x) = 3x + 5$

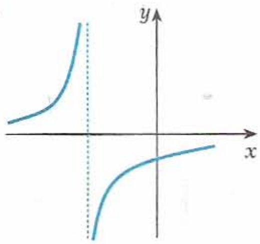
e. $f(x) = \dfrac{x-1}{x-2}$

11. State whether the following graphs belong to one-to-one functions or not.

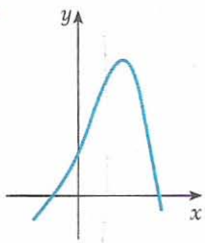
a.



b.



c.



12. Find the inverses of the following one-to-one functions:

a. $f(x) = 6 - 4x$

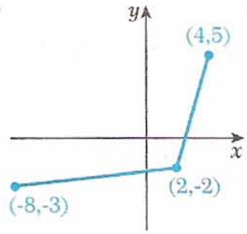
b. $f(x) = \dfrac{3x+4}{x-1}$

c. $f(x) = 4x^3 - 5$

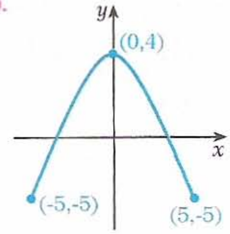
d. $f(x) = 3 \cdot \sqrt[5]{x-1} + 9$

13. Plot the graphs of inverse functions for the following functions, if possible:

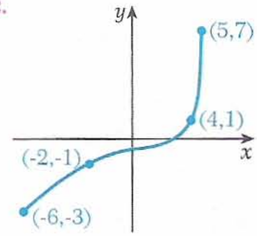
a.



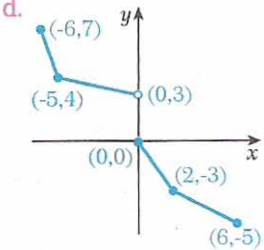
b.



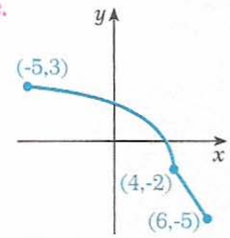
c.



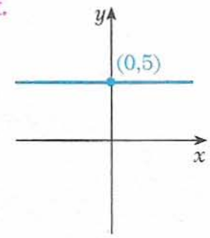
d.



e.



f.



14. Find the required values using the given data for the functions with the inverse.

a. $f(x) = \dfrac{2x+5}{5},\ f^{-1}(3) = ?$

b. $f^{-1}(x) = x^2 - 3,\ D(f^{-1}) = [0, \infty),\ f(-3) = ?$

c. $2f(x) - 1 = \dfrac{f(x)+1}{x-2},\ f^{-1}(2) = ?$

d. $f(3x+1) = \dfrac{4x-1}{x-2},\ f^{-1}(3) = ?$

e. $f(x^2 + x + 1) = x - 2,\ D(f) = \left[\dfrac{3}{4}, \infty\right),$
$f^{-1}(0) = ?$

15. Find the following using the given data for functions with inverses:

a. $f\left(\dfrac{2x+1}{x-1}\right)=x+1,\ f^{-1}(x)=?$

b. $f(x-1)=\dfrac{3x-2}{x+21},\ f^{-1}(x)=?$

c. $f\left(\dfrac{3x-5}{x+1}\right)=x,\ f^{-1}(2x)=?$

Mixed Problems

16. Find the following using the given data for functions with inverses:

a. $f(x) = 3x + 1,\ g(x) = 2x + 3,\ (f \circ g^{-1})(x) = ?$

b. $f(x)=\dfrac{1}{x},\ g(x)=\dfrac{x+2}{2x+1},\ (g^{-1} \circ f)(x)=?$

c. $f^{-1}(x)=\dfrac{2x+1}{x-3},\ g^{-1}(x)=\dfrac{3x+1}{x-2},\ (f^{-1} \circ g^{-1})(x)=?$

d. $f^{-1}(x)=\dfrac{x-2}{x},\ g^{-1}(x)=\dfrac{2}{x+2},\ (f \circ g)^{-1}(x)=?$

e. $f(x) = 2x + 3,\ (g \circ f)(x) = 4x - 5,\ g(x) = ?$

f. $g(x)=\dfrac{3x+1}{x-1},\ (f \circ g)(x)=2x+3,\ f^{-1}(x)=?$

g. $g(x)=\dfrac{2x+1}{x},\ (g^{-1} \circ f)(x)=2x-1,\ f^{-1}(x)=?$

17. Find the required values using the given data for functions with inverses:

a. $f(x) = x + 1,\ g(x) = 3x - 4,\ (f^{-1} \circ g)(2) = ?$

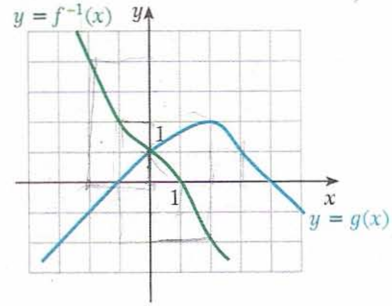
b. $f(x) = x^2 - 2,\ g(x) = 2x + 1,\ (f \circ g^{-1})(3) = ?$

c. $f(x)=\dfrac{x+3}{x-2},\ g(x)=6x+1,\ (f \circ g)^{-1}(2)=?$

d. $f\left(\dfrac{2x+3}{x+1}\right)=x,\ g^{-1}\left(\dfrac{2x+5}{x+3}\right)=3x-2,$

$(f \circ g^{-1})(3)=?$

18. Find the following using the graph of functions below:



a. $(g \circ f^{-1})(-2) + (f^{-1} \circ g \circ f)(2)$

b. $(f^{-1} \circ g)(2) + (f \circ g)(3)$

19. Prove that

a. $f(x) = x^2 - 2x$ is not one-to-one.

b. $f(x) = 2x^3 + 4$ is one-to-one.

20. If f and g are even functions such that

$f(0) = g(4) = -2,\ g(0) = f(4) = 4,$

$f(-2) = g(-2) = 0$, find

$\dfrac{(g \circ f)(4)+(f \circ g)(-4)+(g \circ f)(-4)}{(f \circ g)(2)+(g \circ f)(2)+(f \circ g)(4)}.$

21. Prove that if f and g are even, then $f + g, f - g, fg, f / g$ are also even.

22. Prove that if f and g are odd, then $f + g, f - g$ are odd and fg is even.

23. Find the inverses of the following functions:

a. $f(x) = x^2 + 6x,\ D(f) = (-\infty, -3]$

b. $f(x) = x^2 - 8x + 5,\ D(f) = [4, \infty)$

24. Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

CRYPTOGRAPHY

Cryptography is the art of secret writing. Its aim is to protect information by transforming it into a coded language which unwanted eyes are unable to use. This transformation, however, must be done in a reversible way so that individuals intended to view the information may do so.

Some of the classical methods of cryptography are as follows (However, none of them are used today, because they are considered either insecure or impractical).

Simple Substitution:

When we encode the message MATH IS COOL, it becomes NBUIAJTADPPM. Here, the rule is that every letter is substituted by the letter that follows it. That is, A by B, B by C, "space" by A, etc.

Mixing:

Take a key sequence consisting of the first few natural numbers in mixed order, for example 3, 5, 1, 2, 4. The rule is that the first letter will move to the 3rd place, the second to 5th place, the third to 1st place, the fourth to 2nd place, the fifth to 4th place, and so on for the next five letters.

So MATH IS COOL becomes THM A CIOS OL.

Note that the "space" character is also treated as a letter.

Method of Vigenere:

Pair each letter with a number with its order in the alphabet as follows:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

Our message MATH IS COOL becomes 13 1 20 8 27 9 19 27 3 15 15 12

Choose a key word, for example, FUNCTION which is 6 21 14 3 20 9 15 14.

Now, add 6 to the first letter of our encoded message, 21 to the second, 14 to the third, and so on.

The new message will be read as 19 22 34 11 47 18 34 41 9 36 29 15.

If the number is more than 27, divide it by 27 and take the remainding number. Finally, the message will become SVGKTRGNIIBO. Find the pair for these numbers from the table above. The whole procedure is summarized in the table below.

13	1	20	8	27	9	19	27	3	15	15	12	*convert the message using table.*
6	21	14	3	20	9	15	14	6	21	14	3	*convert the keyword and write it as the same length as the original message.*
19	22	34	11	47	18	34	41	9	36	29	15	*add the top two rows.*
19	22	7	11	20	18	7	14	9	9	2	15	*divide the previous row by 27 and write the remainding number.*

Note that all of these methods are functions that are defined from a limited domain to a range. We can invent even more complicated methods by applying a few of them together for the same message (Think about the composition of functions!). Here it is very important to know the rules. To decode a crypted message we just apply the rule in reverse order (Think about the inverse of a function!).

Below is an encoded message. We first applied the Method of Vigenere with the key word CAT (each letter is paired by its order in the alphabet as "space" being the 27th letter), and then we applied the simple substitution.

GQHKTVXWFEVCSPM

Try to decode it!

What will happen if the rule is a function that has no inverse? What happens if we assign the same number to two different letters? If a few of the rules are applied in order, by which rule should we start the decoding procedure? Try to develop your own method of encryption.

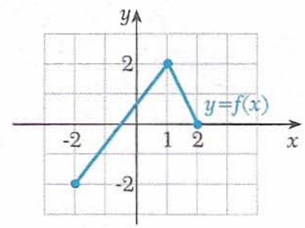
4 PLOTTING GRAPHS OF FUNCTIONS

A. TRANSFORMATION OF GRAPHS

To have a better understanding of how to plot a graph of a function it is very important to know how certain transformations affect its graph. The transformations we study will be shifting, reflecting, and stretching.

1. Vertical Shift

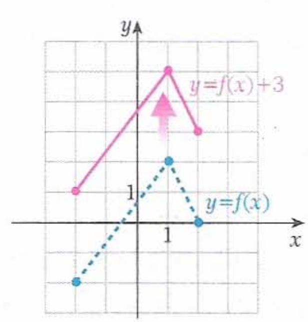
Consider the function $y = f(x)$ whose graph is given below:



Here, it is clear that $f(-2) = -2$. Now let us consider another function $y = f(x) + 3$. When $x = -2$, the value of $y = f(x) + 3$ will be $y = f(-2) + 3 = -2 + 3 = 1$. So for each x-value, the y-value of those functions differ by 3. We can illustrate this fact when x is equal to -2, 1 and 2 in the following table:

x	$y = f(x)$	$y = f(x) + 3$
-2	-2	1
1	2	5
2	0	3

Clearly for the same x-value, the y-coordinate of the graph of the second function is 3 units higher than the y-coordinate of the first function. That is, the graph of the second function is a "3 unit shifted up" version of the first one. Using this principle we can plot the graph of $y = f(x) + 3$ as below :



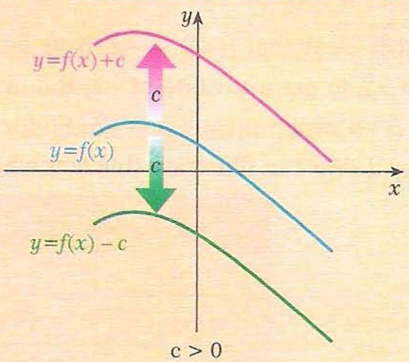
Note

Each point (x, y) on the graph of $y = f(x)$ is translated to $(x, y + c)$ on the graph of $y = f(x) + c$, where $c \in \mathbb{R}$.

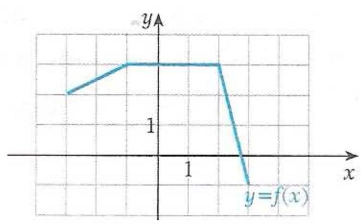
In general, given the graph of $y = f(x)$ and $c > 0$, we obtain the graph of

1. $y = f(x) + c$ by shifting the graph of $y = f(x)$ upward c units,
2. $y = f(x) - c$ by shifting the graph of $y = f(x)$ downward c units.

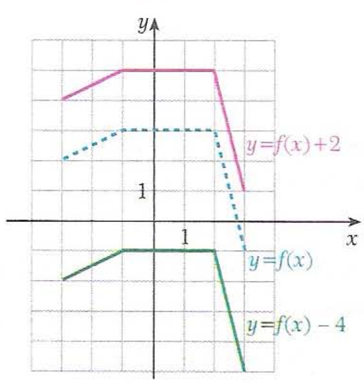
VERTICAL SHIFT OF THE GRAPHS



Example 80 Given the graph of $f(x)$, plot the graph of $f(x) + 2$ and $f(x) - 4$.

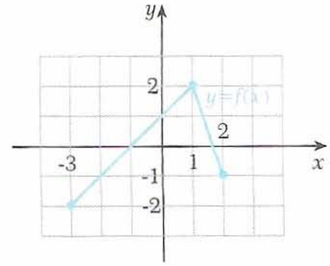


Solution Clearly, to obtain the graph of $f(x) + 2$ we shift the graph of $f(x)$ upward 2 units and to obtain the graph of $f(x) - 4$ we shift the graph of $f(x)$ downward 4 units.



2. Horizontal Shift

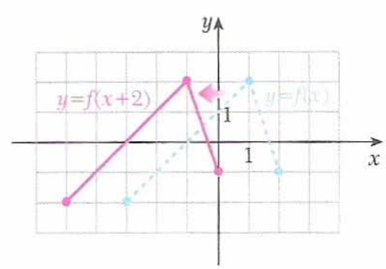
Consider the function $y = f(x)$ whose graph is given below:



Here, it is clear that $f(-3) = -2$. Now let us consider another function $y = f(x + 2)$. Since $f(-3) = -2$ to get a y-value of -2, the argument of $f(x + 2)$ should be equal to -3, that is, $x + 2 = -3$ which means $x = -5$. So for each given y-value, the x-value of those functions differ by 2. We can illustrate this fact when y is equal to -2, -1 and 2 in the following table:

y	x value for $f(x)$	x value for $f(x + 2)$
-2	-3	-5
-1	2	0
2	1	-1

Clearly for the same y-value, the x-coordinate of the graph of the second function is 2 units left with respect to the x-coordinate of the first function. That is, the second function is a "2 units shifted left" version of the first one. Using this principle we can plot the graph of $y = f(x + 2)$ as below:



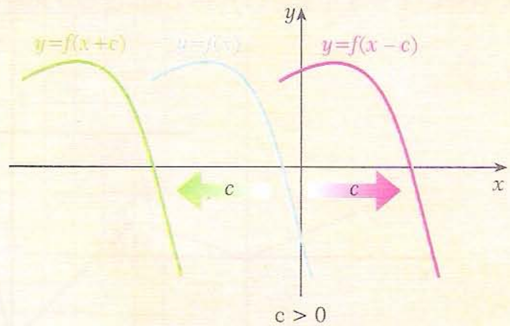
Note

Each point (x, y) on the graph of $y = f(x)$ is translated to $(x - c, y)$ on the graph of $y = f(x + c)$, where $c \in \mathbb{R}$.

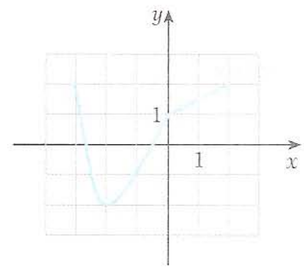
In general, given the graph of $y = f(x)$ and $c > 0$, we obtain the graph of

1. $y = f(x + c)$ by shifting the graph of $y = f(x)$ to the left c units,
2. $y = f(x - c)$ by shifting the graph of $y = f(x)$ to the right c units.

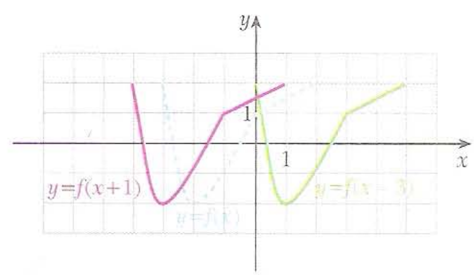
HORIZONTAL SHIFT OF THE GRAPHS



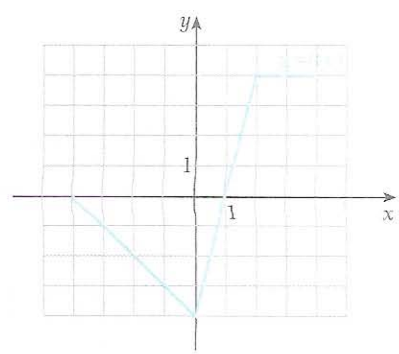
Example 81 Given the graph of $f(x)$, plot the graph of $f(x-3)$ and $f(x+1)$.



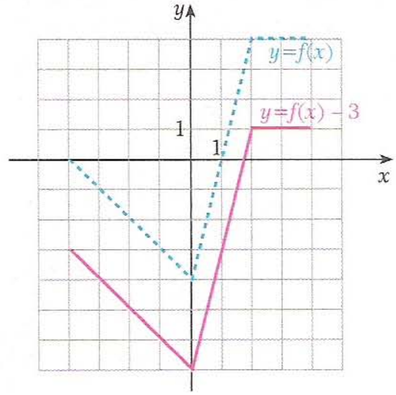
Solution Clearly, to obtain the graph of $f(x-3)$ we shift the graph of $f(x)$ to the right 3 units and to obtain the graph of $f(x+1)$ we shift the graph of $f(x)$ to the left 1 unit.



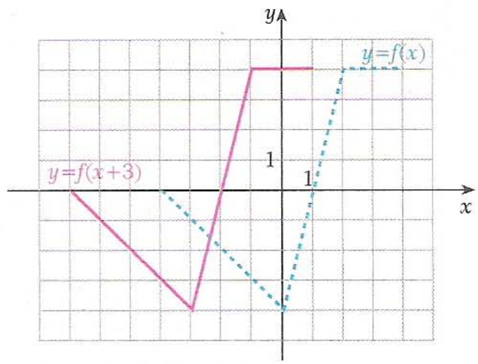
Example 82 Given the graph of $f(x)$, plot the graphs of $f(x)-3$, $f(x+3)$ and $f(x-4)+2$.



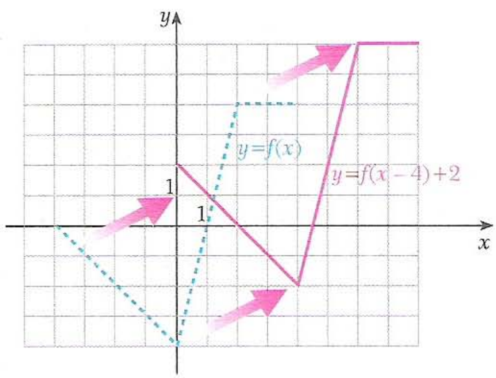
Solution The graph of $f(x) - 3$ is obtained by shifting the graph of $f(x)$ downward 3 units:



The graph of $f(x + 3)$ is obtained by shifting the graph of $f(x)$ to the left 3 units:



The graph of $f(x - 4) + 2$ is obtained by the combination of shifting the graph of $f(x)$ to the right 4 units and upward 2 units:

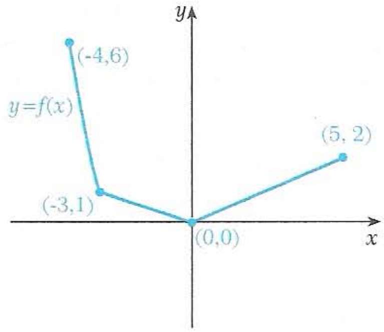


Note

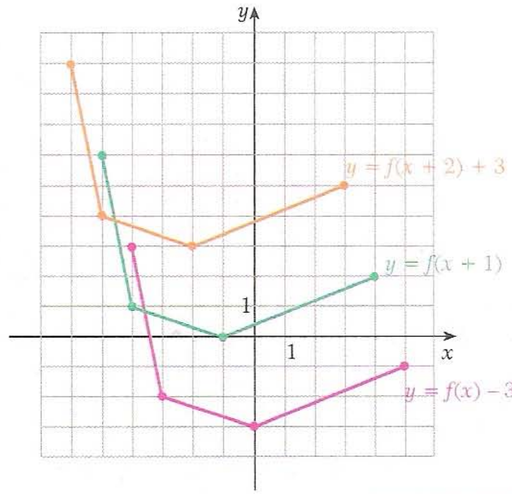
Given the graph of $f(x)$ and $a, b \in \mathbb{R}$, to plot the graph of $f(x + a) + b$, we apply the horizontal and the vertical shift together in any order.

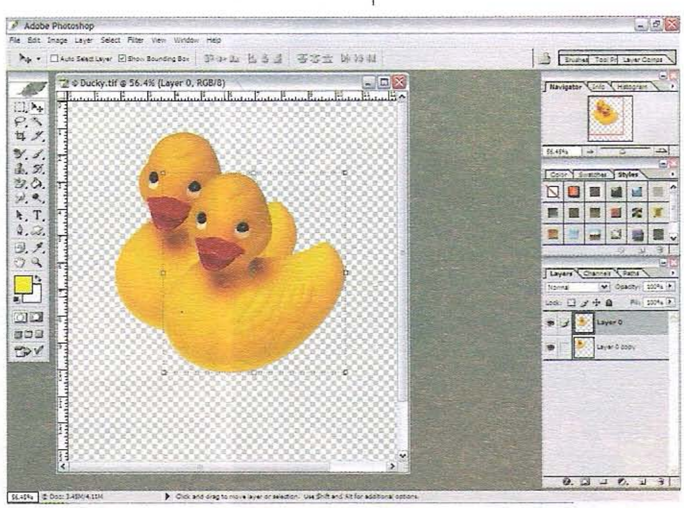
Check Yourself 17

Given the graph of $f(x)$, plot the graphs of $f(x) - 3$, $f(x + 1)$ and $f(x + 2) + 3$.



Answers





Translation is a very powerful tool in image editing software.

3. Reflection

We can use the graph of $y = f(x)$ to obtain the graphs of $y = -f(x)$ and $y = f(-x)$. The y-coordinate of each point on the graph of $y = -f(x)$ is simply the negative of the y-coordinate of the corresponding x-coordinate on the graph of $y = f(x)$. So the graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ in x-axis. On the other hand, the value of $y = f(-x)$ at $x = x_0$ is the same as the value of $y = f(x)$ at $x = -x_0$. So the graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ in the y-axis.

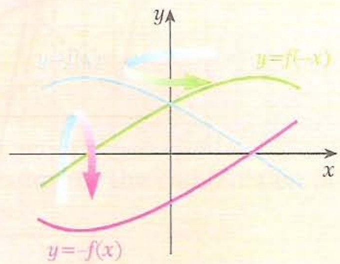
Note

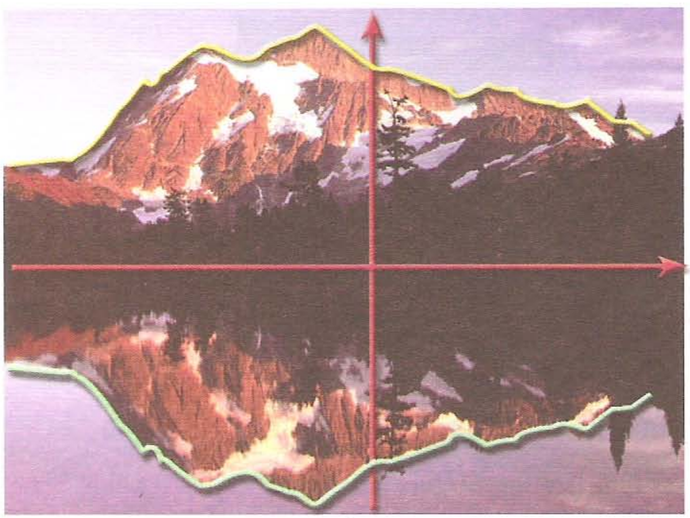
1. Each point (x, y) on the graph of $y = f(x)$ is translated to $(x, -y)$ on the graph of $y = -f(x)$.
2. Each point (x, y) on the graph of $y = f(x)$ is translated to $(-x, y)$ on the graph of $y = f(-x)$.

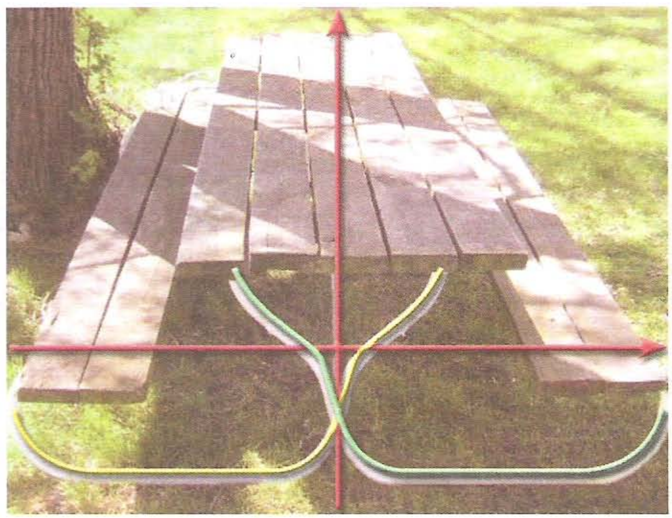
In general, given the graph of $y = f(x)$, we obtain the graph of

1. $y = -f(x)$ by multipying each y-coordinate by -1,
2. $y = f(-x)$ by multiplying each x-coordinate by -1.

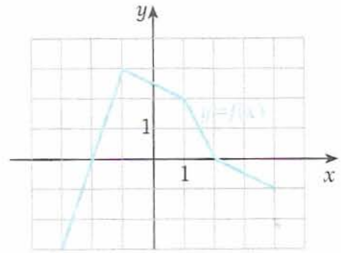
REFLECTION OF THE GRAPHS



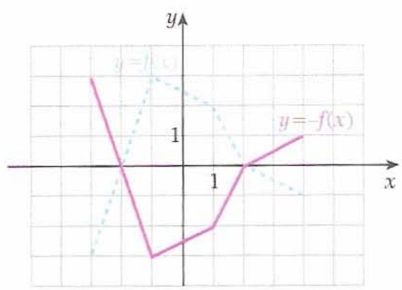


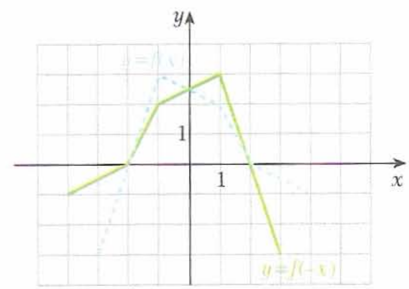


Example 83 Given the graph of $f(x)$, plot the graphs of $-f(x)$ and $f(-x)$.

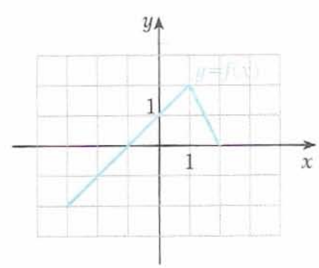


Solution The graph of $-f(x)$ is obtained by reflecting the graph of $f(x)$ in the x-axis and the graph of $f(-x)$ is obtained by reflecting the graph of $f(x)$ in the y-axis:

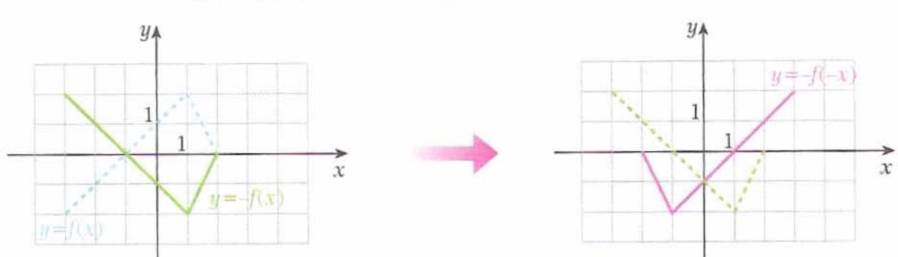




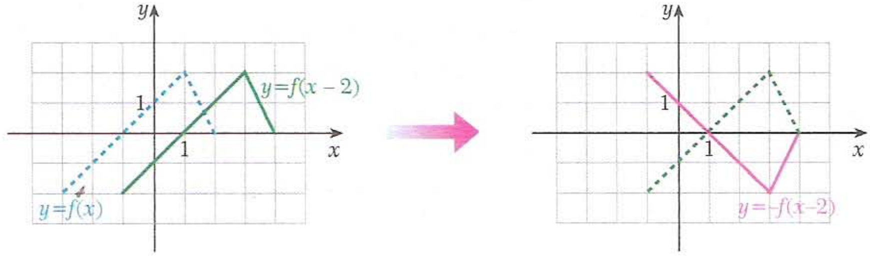
Example 84 Given the graph of $f(x)$, plot the graphs of $-f(-x)$, $-f(x-2)$ and $f(-x)+1$.



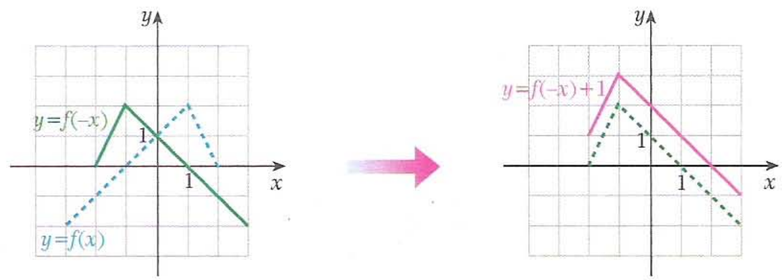
Solution The graph of $-f(-x)$ is obtained by reflecting the graph of $f(x)$ in the x and y axes together in any order. Here we first plot $-f(x)$ and then $-f(-x)$:



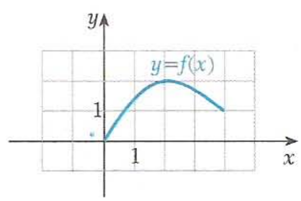
To obtain the graph of $-f(x-2)$ from $f(x)$, we first plot $f(x-2)$ by the shift of $f(x)$ to the right 2 units and then $-f(x-2)$ by the reflection of $f(x-2)$ in the x-axis.



To obtain the graph of $f(-x)+1$ from $f(x)$, we first plot $f(-x)$ by the reflection of $f(x)$ in the y-axis and then $f(-x)+1$ by the shift of $f(-x)$ upward 1 unit.



Example 85 Given the graph of $f(x)$, plot the graph of $-f(-x+2)+3$.



Solution Note that this example requires the combination of a horizontal and a vertical reflection together with a horizontal and a vertical shift. To plot the correct graph it is very important to apply those transformations in the correct order. Otherwise we may be faced with the wrong graph. As a rule we should always deal with the horizontal transformations and the vertical transformations apart from each other. Although it does not matter by which one we begin, the order between a shift and reflection critically matters.

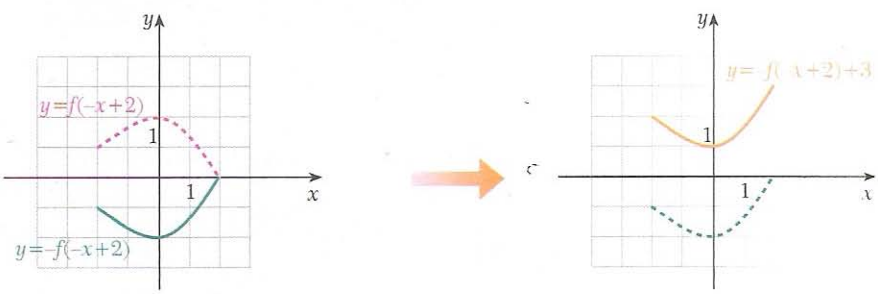
Let us deal with the horizontal transformations (the expression inside the brackets of f) initially. That is, first of all we plot $f(-x+2)$ which requires one horizontal reflection and one horizontal shift. Clearly, it will be wrong if we first plot $f(x+2)$ by horizontal shift and then take reflection in x-axis since this will result in graph of $f(-(x+2)) = f(x-2)$, not $f(-x+2)$.

So we should first deal with reflection. But it will also be wrong to plot $f(-x)$ and then get $f(-x + 2)$ by shifting to the left two units.

It is easy to convince ourselves why that will be wrong: On the given graph we can see that $(4, 1)$ is a point on $y = f(x)$. For which value of x will $f(-x + 2) = 1$ be correct? Of course when $-x + 2 = 4$. That makes -2 the value of x which makes the equation correct. So point $(4, 1)$ of $y = f(x)$ will translate to point $(-2, 1)$ of $f(-x + 2)$. Now let us assume that plotting $f(-x)$ and then getting $f(-x + 2)$ by shifting the previous one to the left two units is correct. Then point $(4, 1)$ of $y = f(x)$ will first move to $(-4, 1)$ of $y = f(-x)$ (reflection in y-axis) and then $(-6, 1)$ of $f(-x + 2)$ (shift to the left two units). But we know that $(4, 1)$ should translate to $(-2, 1)$, not $(-6, 1)$!

So how to apply the correct order? The problem with the above idea is that we did not consider the fact that the coefficient of x also affects the horizontal shift. To plot $f(-x + 2)$ we should first write it in the form $f(-(x - 2))$. Then we plot $f(-x)$ by a reflection in y-axis and $f(-(x - 2)) = f(-x + 2)$ by shifting it two units to the right. So the graph of $f(-x + 2)$ will be as below. Note that $(-4, 1)$ is really translated to $(-2, 1)$.

Now we will deal with the vertical transformations (the expression outside the brackets of f). That is, we plot $-f(-x + 2) + 3$ which requires one vertical reflection and one vertical shift. For each y-coordinate of $y = f(-x + 2)$, new y-coordinate becomes $-y + 3$. Algebraically we first multiply each y-coordinate by -1 and then add 3 to the result. Graphically we first take a reflection in the x-axis and then shift the graph upward three units. So, first we plot $-f(-x + 2)$ and then $-f(-x + 2) + 3$ as below:



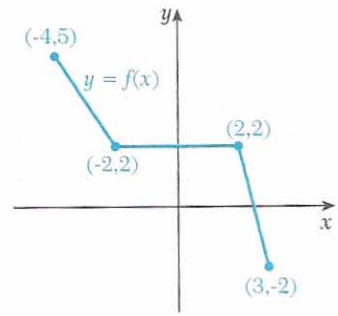
Note

Given the graph of $f(x)$ and $a, b \in \mathbb{R}$, to plot the graph of $-f(-x + a) + b$ we apply the following transformation order:

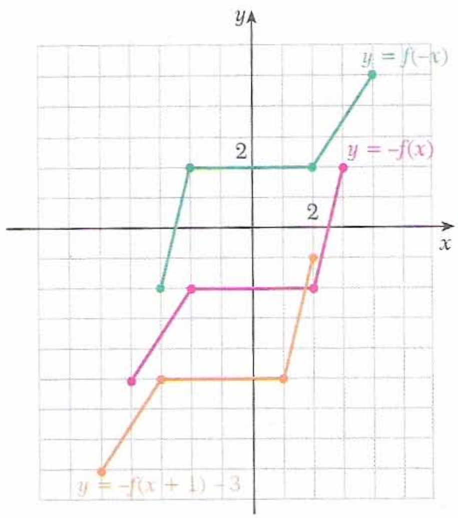
1. Plot $f(-x)$.
2. Plot $f(-(x - a))$.
3. Plot $-f(-x + a)$.
4. Plot $-f(-x + a) + b$.

Check Yourself 18

Given the graph of $f(x)$, plot $-f(x)$, $f(-x)$ and $-f(x+1)-3$.



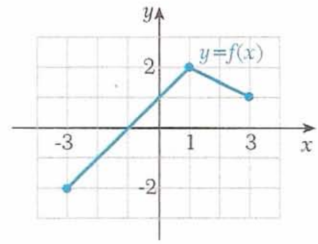
Answers



Symmetry as a result of reflection is widely used in carpets.

4. Vertical Stretch and Shrink

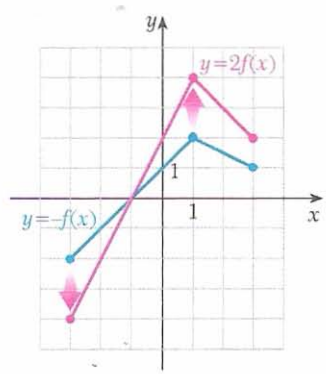
Consider the function $y = f(x)$ whose graph is given below:



Here, it is clear that $f(1) = 2$. Now let us consider another function $y = 2f(x)$. When $x = 1$, the value of $y = 2f(x)$ will be $y = 2 \cdot f(1) = 2 \cdot 2 = 4$. So for each x-value, the y-value of the first function will be multiplied by 2. We can illustrate this fact when x is equal to –3, 1 and 3 in the following table:

x	$y = f(x)$	$y = 2f(x)$
–3	–2	–4
1	2	4
3	1	2

Clearly for the same x-value, the y-coordinate of the graph of the second function is twice as big as the y-coordinate of the first function. That is, the graph of the second function is a "twice vertical stretched" version of the first one. Using this principle we can plot the graph of $y = 2f(x)$ as below:

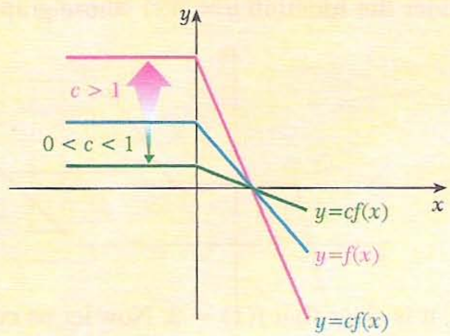


Note

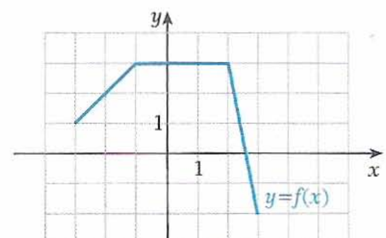
Each point (x,y) on the graph of $y = f(x)$ is translated to (x,cy) on the graph of $y = cf(x)$, where $c \in \mathbb{R}$.

In general, given the graph of $y = f(x)$, we obtain the graph of $y = cf(x)$ by stretching $(c > 1)$ or shrinking $(0 < c < 1)$ the graph of $y = f(x)$ vertically by a factor of c. Keep in mind that for negative c values we have to apply an additional reflection with respect to the x-axis. In short, it is enough to multiply each y-coordinate by c and keep the x-coordinates without change.

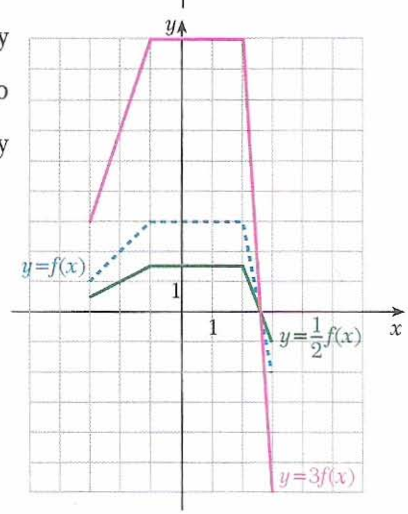
VERTICAL STRETCH AND SHRINK OF THE GRAPHS



Example 86 Given the graph of $f(x)$, plot the graph of $3f(x)$ and $\frac{1}{2}f(x)$.

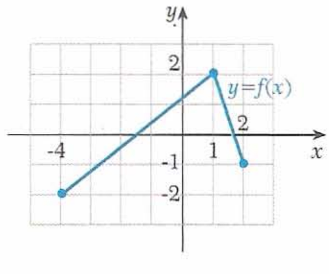


Solution To obtain the graph of $3f(x)$, we stretch the graph of $f(x)$ by a factor of 3 (multiply each y-coordinate by 3) and to obtain the graph of $\frac{1}{2}f(x)$, we shrink the graph of $f(x)$ by a factor of 2 (multiply each y-coordinate by $\frac{1}{2}$).



5. Horizontal Stretch and Shrink

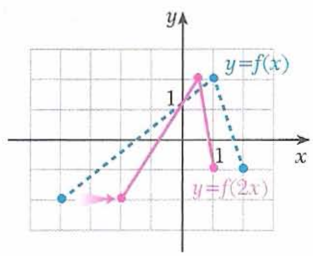
Consider the function $y = f(x)$ whose graph is given below:



Here, it is clear that $f(2) = -1$. Now let us consider another function $y = f(2x)$. Since $f(2) = -1$ to get a y-value of -1, the argument of $f(2x)$ should be equal to 2, that is, $2x = 2$ which means $x = 1$. So for each given y-value, the x-value of the first function will be divided by 2. We can illustrate this fact when y is equal to -2, -1 and 2 in the following table:

y	x value for $f(x)$	x value for $f(2x)$
-2	-4	-2
-1	2	1
2	1	0.5

Clearly for the same y-value, the x-coordinate of the graph of the second function is half of the x-coordinate of the first function. That is, the second function is a "half horizontal shrinked" version of the first one. Using this principle we can plot the graph of $y = f(2x)$ as below:

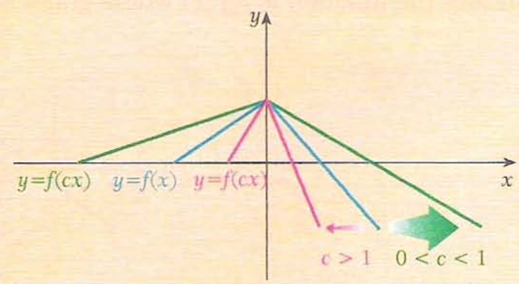


Note

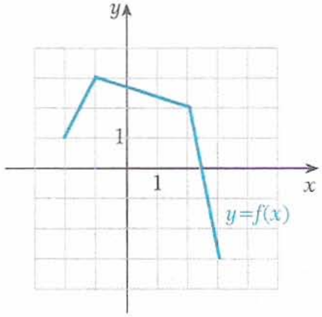
Each point (x,y) on the graph of $y = f(x)$ is translated to $(\frac{x}{c}, y)$ on the graph of $y = f(cx)$, where $c \in \mathbb{R}$.

In general, given the graph of $y = f(x)$, we obtain the graph of $y = f(cx)$ by stretching $(0 < c < 1)$ or shrinking $(c > 1)$ the graph of $y = f(x)$ horizontally by a factor of c. Keep in mind that for negative c values we have to apply an additional reflection with respect to the y-axis. In short, it is enough to divide each x-coordinate by c and keep the y-coordinates without change.

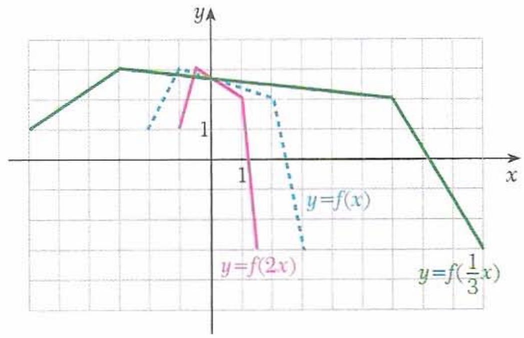
HORIZONTAL STRETCH AND SHRINK OF THE GRAPHS



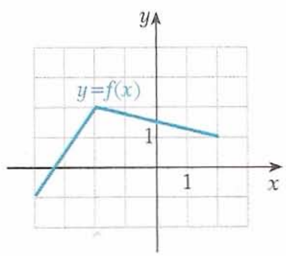
Example 87 Given the graph of $f(x)$, plot the graph of $f(2x)$ and $f(\frac{1}{3}x)$.



Solution To obtain the graph of $f(2x)$ we shrink the graph of $f(x)$ by a factor of 2 (divide each x-coordinate by 2) and to obtain the graph of $f(\frac{1}{3}x)$ we stretch the graph of $f(x)$ by a factor of 3 (divide each x-coordinate by $\frac{1}{3}$):



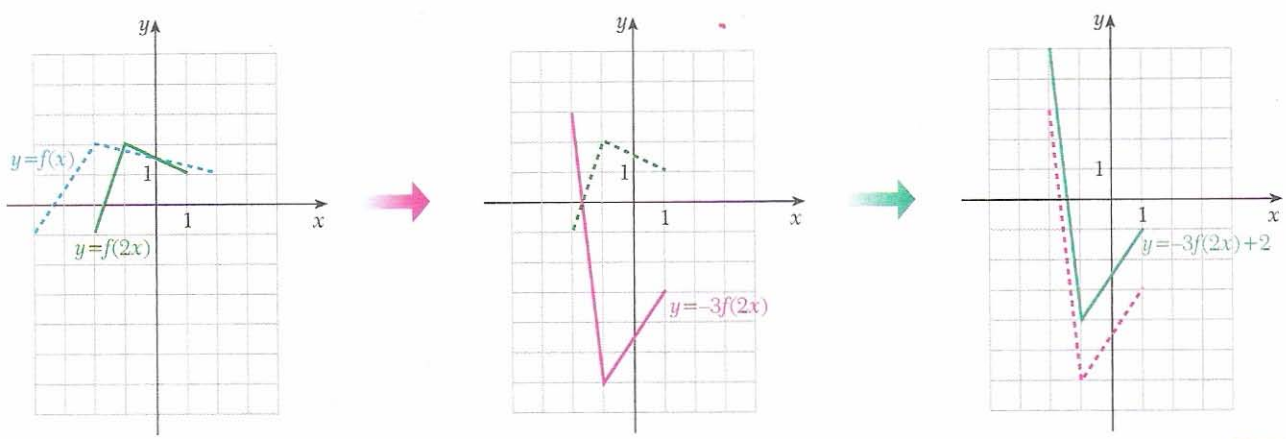
Example 88 Given the graph of $f(x)$, plot the graph of $-3f(2x) + 2$.



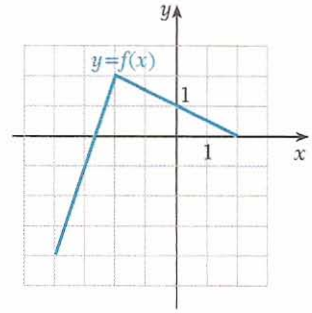
Solution To obtain the graph of $-3f(2x) + 2$ we apply the following procedure:

1. Plot $f(2x)$ by horizontal shrink (divide each x-coordinate by 2).
2. Plot $-3f(2x)$ by vertical stretch with reflection (multiply each y-coordinate by -3).
3. Plot $-3f(2x) + 2$ by vertical shift (add 2 to each y-coordinate).

The resulting graph is as follows:

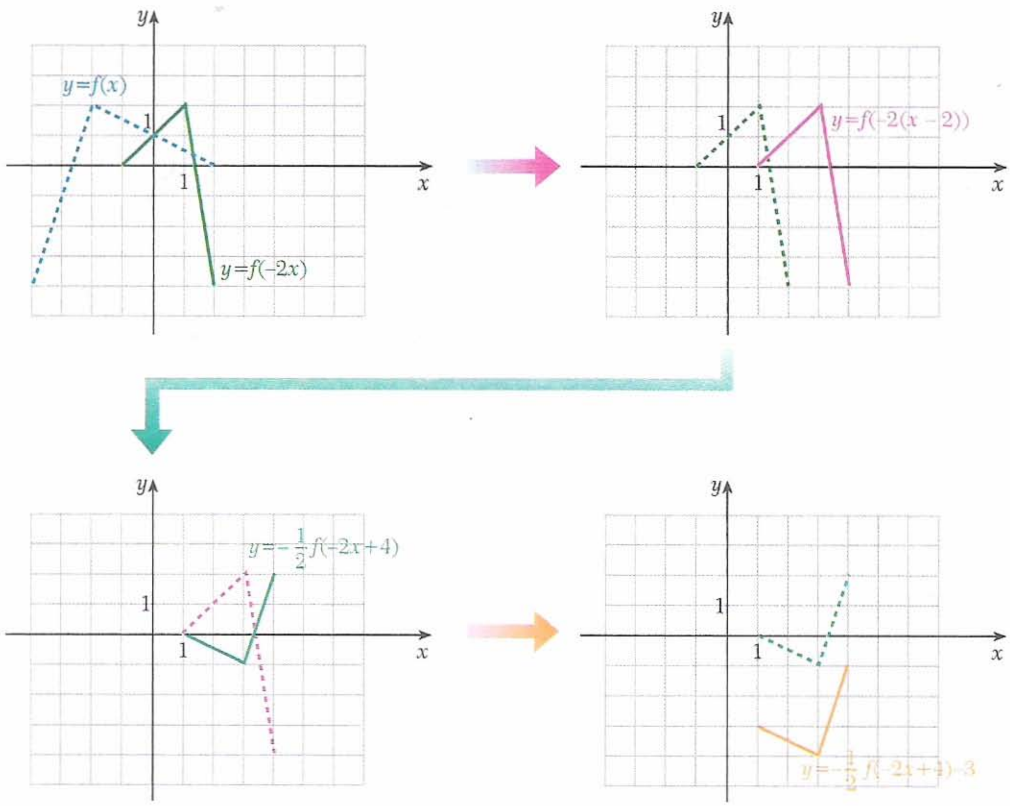


Example 89 Given the graph of $f(x)$, plot the graph of $-\frac{1}{2}f(-2x+4)-3$.



Solution 1 This example contains all the transformations we have learned. As we learned from example 85, applying the correct order of the transformation is very important to plot the right graph. The procedure is as follows:

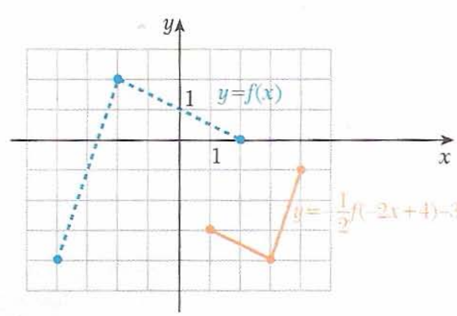
1. Plot $f(-2x)$ by horizontal shrink with reflection (divide each x-coordinate by -2).
2. Plot $f(-2(x-2))$ by horizontal shift (add 2 to each x-coordinate).
3. Plot $-\frac{1}{2}f(-2x+4)$ by vertical shrink with reflection (multiply each y-coordinate by $-\frac{1}{2}$).
4. Plot $-\frac{1}{2}f(-2x+4)-3$ by vertical shift (subtract 3 from each y-coordinate).



Solution 2 In the previous solution we plotted four different graphs to obtain the desired graph. An alternative solution is considering the final locations of a few "key" points on the given graph and then connecting these new points to plot the final graph. There are three key points on the graph of $y = f(x)$: $(-4, -4)$, $(-2, 2)$, $(2, 0)$. let us find out where the point $(2, 0)$ of $y = f(x)$ will be translated into $y = -\frac{1}{2}f(-2x + 4) - 3$. Here the coefficients that are out of f-brackets affect the y-value, so the y-coordinate of the point $(2, 0)$ will move to $-\frac{1}{2} \cdot 0 - 3 = -3$. Clearly we assumed that $f(-2x + 4) = 0$. Since the point $(2, 0)$ is on the graph of $y = f(x)$, we have $f(2) = 0$. So $-2x + 4 = 2$, that is $x = 1$. So the point $(2, 0)$ will move to $(1, -3)$. The procedure can be summarized as follows:

Old point	To find the new x-coordinate solve	The new x-coordinate	To find the new y-coordinate calculate $-\frac{1}{2}f(-2x+4)-3$ such that	The new y-coordinate
$(-4, -4)$	$-2x+4 = -4$	4	$f(-2x + 4) = -4$	-1
$(-2, 2)$	$-2x+4 = -2$	3	$f(-2x + 4) = 2$	-4
$(2, 0)$	$-2x+4 = 2$	1	$f(-2x + 4) = 0$	-3

Plotting those new points and connecting them to get the graph will give us the same result as with the previous solution as demonstrated in the graph below:



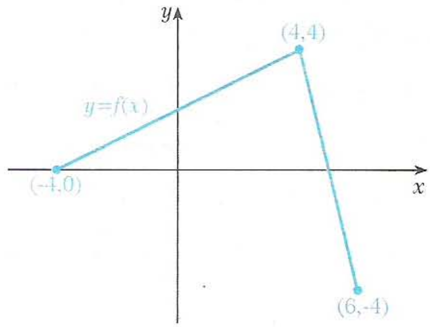
Note

Given the graph of $f(x)$ and $a, b, c, d \in \mathbb{R}$, to plot the graph of $af(bx + c) + d$ we apply the following transformation order:

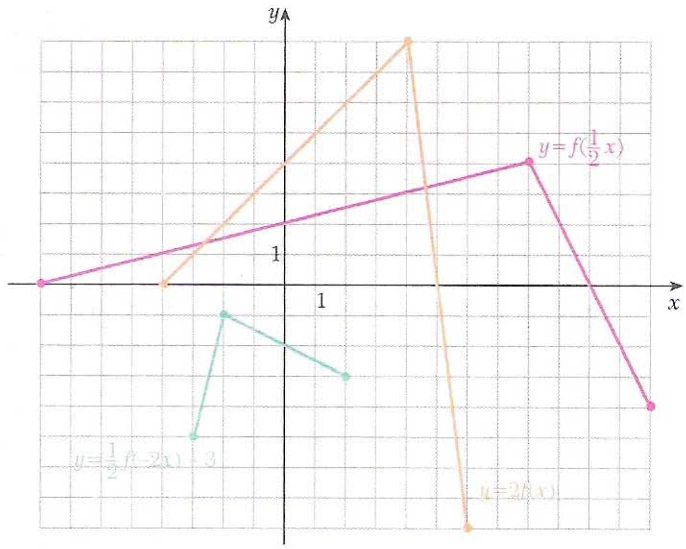
1. Plot $f(bx)$. *(divide each x-coordinate by b)*
2. Plot $f(b(x+\frac{c}{b}))$. *(subtract $\frac{c}{b}$ from each x-coordinate)*
3. Plot $af(bx + c)$. *(multiply each y-coordinate by a)*
4. Plot $af(bx + c) + d$. *(add d to each y-coordinate)*

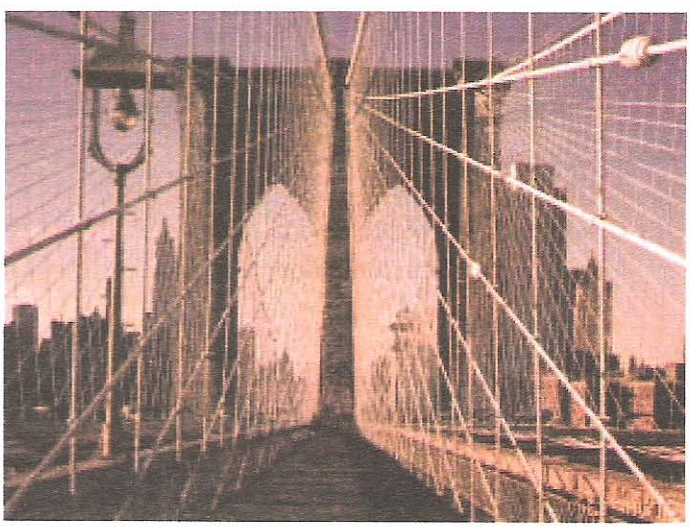
Check Yourself 19

Given the graph of $f(x)$, plot the graphs of $2f(x)$, $f(\frac{1}{2}x)$ and $\frac{1}{2}f(-2x)-3$.



Answers



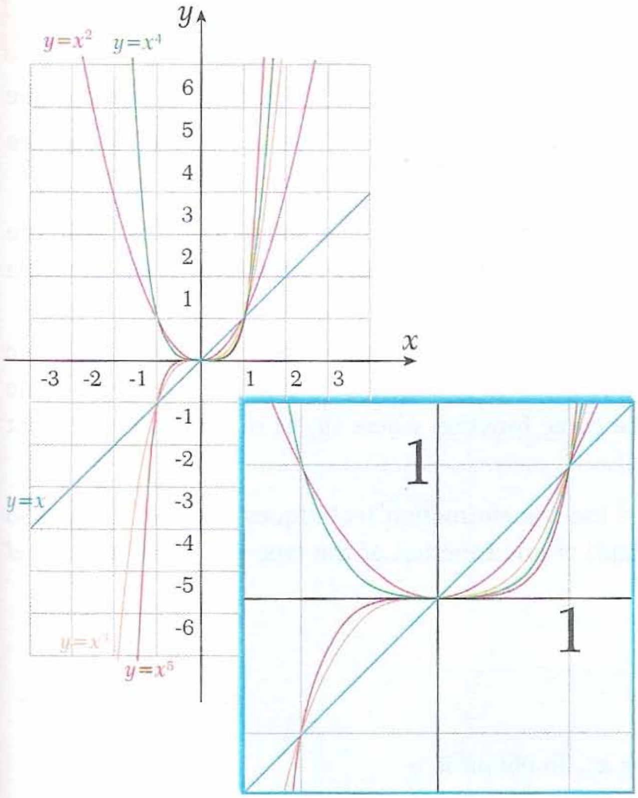


B. GRAPHS OF ELEMENTARY FUNCTIONS

1. Functions of the Form $y = x^n$, $n \in N$

Let us plot the graphs of $y = x$, $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$. Since we do not have any idea about what they look like, we will construct a table of the values for certain x-values.

Using the data from table (below right), the graphs will be constructed as demonstrated below on the left:

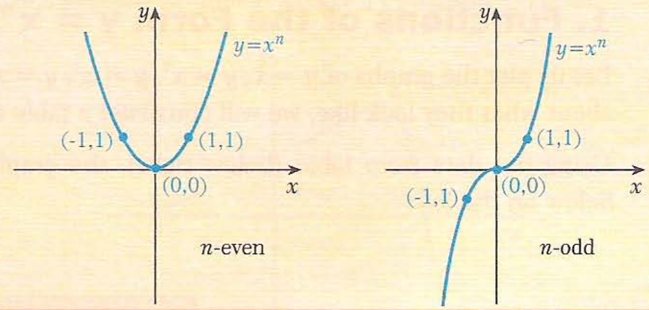


x	$y = x$	$y = x^2$	$y = x^3$	$y = x^4$	$y = x^5$
–2	–2	4	–8	16	–32
–1	–1	1	–1	1	–1
–1/2	–1/2	1/4	–1/8	1/16	–1/32
0	0	0	0	0	0
1/2	1/2	1/4	1/8	1/16	1/32
1	1	1	1	1	1
2	2	4	8	16	32

Investigating the graphs we can derive the following results:

Properties of the function $y = x^n$	
n even	n odd
The domain is $\mathbb{R}$.	The domain is $\mathbb{R}$.
The range is $[0, \infty)$.	The range is $\mathbb{R}$.
The function is even.	The function is odd.
The graph passes through $(-1, 1)$, $(0, 0)$, $(1, 1)$.	The graph passes through $(-1, -1)$, $(0, 0)$, $(1, 1)$.
For $\lvert x \rvert > 1$; the bigger n is, the closer are the arms of the graph to the y-axis.	
For $\lvert x \rvert < 1$; the bigger n is, the closer are the arms of the graph to the x-axis.	

GRAPH OF $y = x^n$, $n \in N$



> The equation $y = ax^2 + bx + c$ gives the general form of a parabola where $\left(-\frac{b}{2a}, y(-\frac{b}{2a})\right)$ is the vertex.

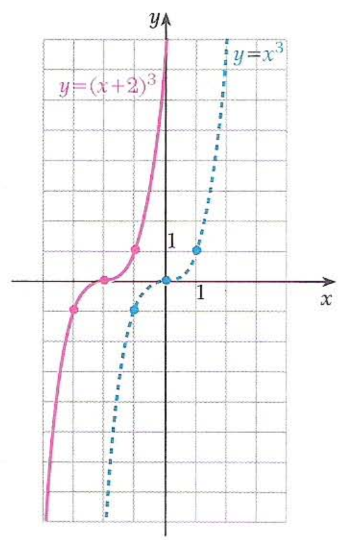
The graph of $y = x$ is called a **line**. The equation $y = mx + n$ is the general equation of the line where m is the **slope**. The slope is defined by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are any two points on the graph of given line.

The graph of $y = x^2$ is called a **parabola**. The point (0, 0) on the graph of $y = x^2$ is called the **vertex** of the parabola. The equation $y = a(x - h)^2 + k$ gives the vertex form of the parabola where (h, k) is the vertex.

The function $y = x^3$ is called the **cubic function**. The point (0, 0) on the graph of the cubic function is the **stationary point of inflection**. The equation $y = a(x - h)^3 + k$ gives the stationary point of inflection form of the cubic function where (h, k) is the stationary point of inflection.

With the help of the above scheme and the transformation techniques we have learned so far, we are now able to plot a rough graph of any function of the type $y = a(bx + c)^n + d$ where $a, b, c, d \in \mathbb{R}$ and $n \in \mathbb{N}$.

Example 90 Plot the graph of $y = (x + 2)^3$.



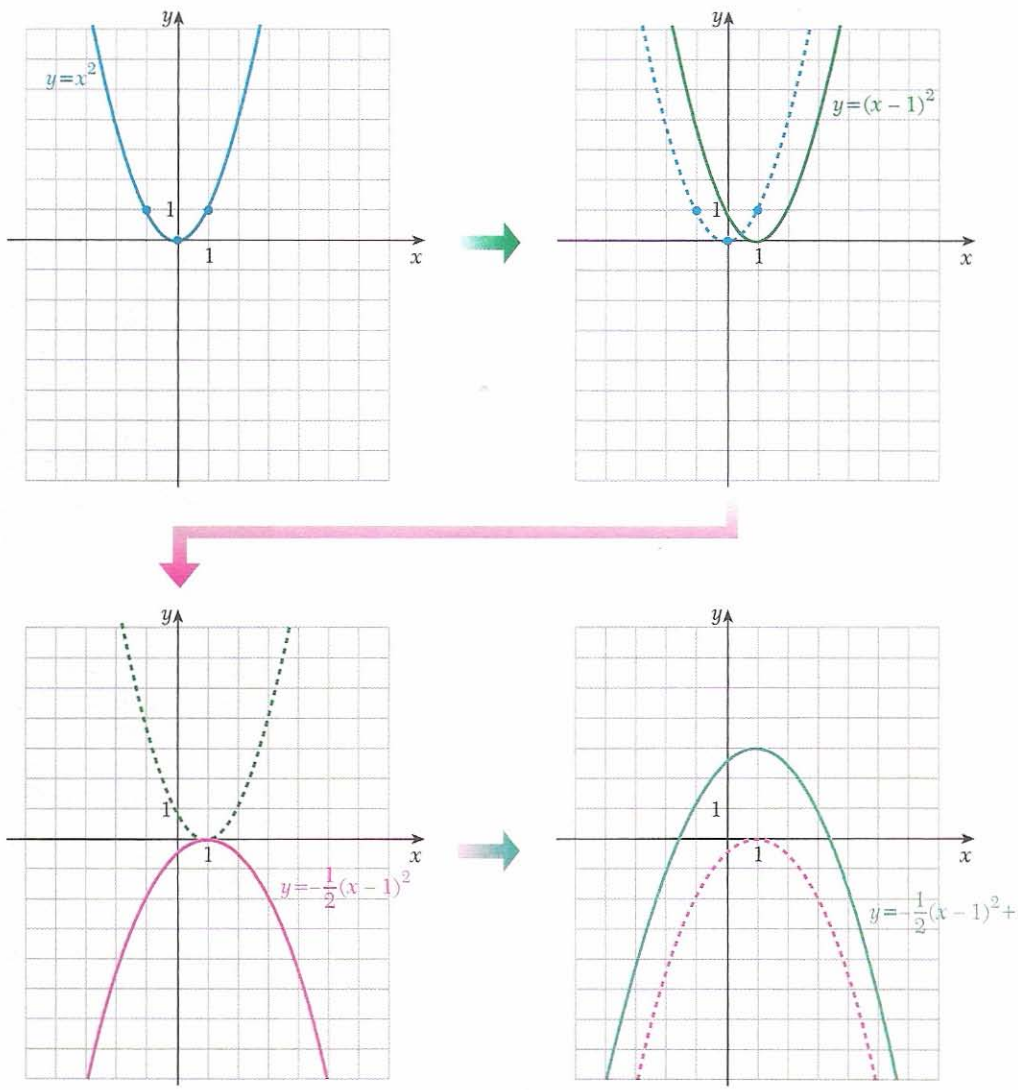
Solution This graph is a transformed form of $y = x^3$. To obtain it we apply the following procedure:

1. Plot $y = x^3$ graphing the common points (–1, –1), (0, 0), (1, 1). We will use these points to help us in transformation. Following new locations of those points will help us to plot the transformed graph.
2. Plot $y = (x + 2)^3$ by a horizontal shift (subtract 2 from each x-coordinate) of $y = x^3$.

Example 91 Plot the graph of $y = -\frac{1}{2}(x-1)^2 + 3$.

Solution This graph is a transformed form of $y = x^2$. To obtain it we apply the following procedure:

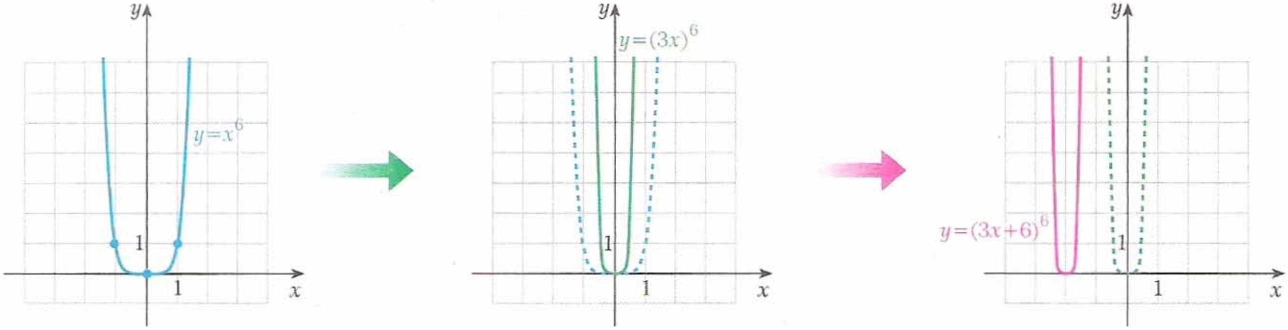
1. Plot $y = x^2$ graphing the common points $(-1, 1)$, $(0, 0)$, $(1, 1)$.
2. Plot $y = (x - 1)^2$ by horizontal shift (add 1 to each x-coordinate).
3. Plot $y = -\frac{1}{2}(x-1)^2$ by vertical shrink (multiply each y-coordinate by $-\frac{1}{2}$).
4. Plot $y = -\frac{1}{2}(x-1)^2 + 3$ by vertical shift (add 3 to each y-coordinate).



Example 92 Plot the graph of $y = (3x + 6)^6$.

Solution This graph is a transformed form of $y = x^6$. To obtain it we apply the following procedure:

1. Plot $y = x^6$ graphing the common points $(-1, 1)$, $(0, 0)$, $(1, 1)$.
2. Plot $y = (3x)^6$ by horizontal shrink (divide each x-coordinate by 3).
3. Plot $y = (3(x + 2))^6 = (3x + 6)^6$ by horizontal shift (subtract 2 from each x-coordinate).

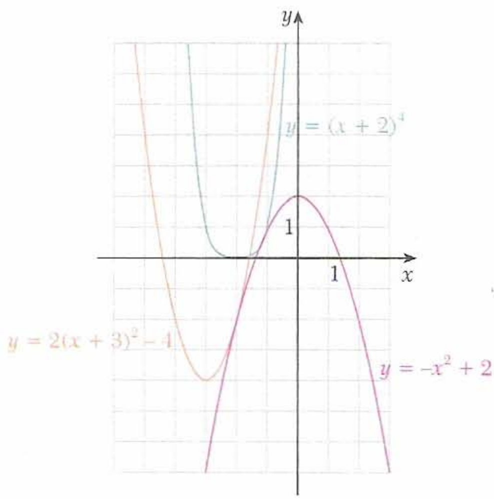


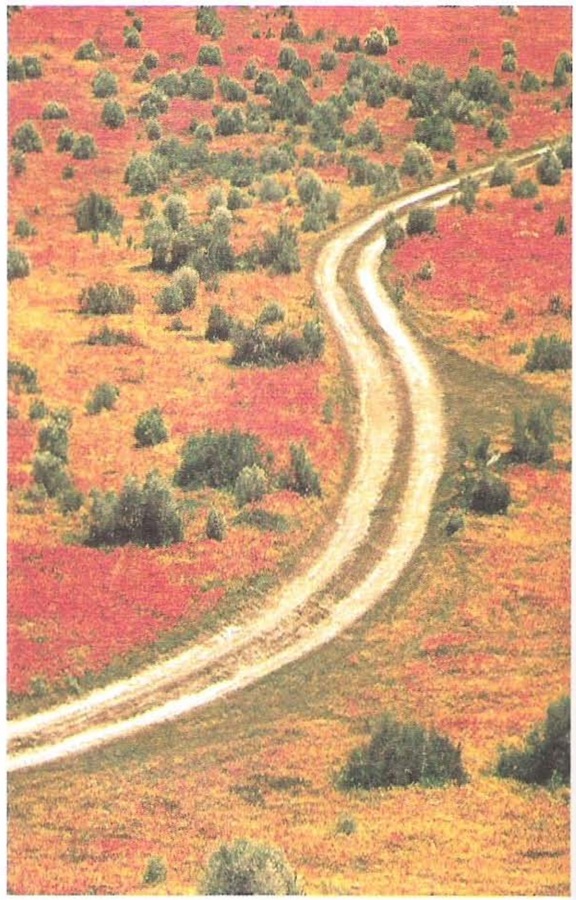
Check Yourself 20

Plot the graph of the following functions:

1. $y = -x^2 + 2$
2. $y = (x + 2)^4$
3. $y = 2(x + 3)^2 - 4$

Answers



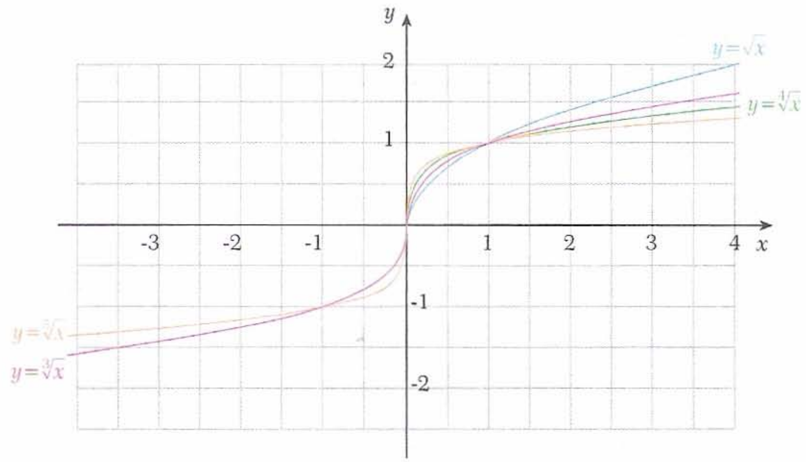


2. Functions of the Form $y = \sqrt[n]{x}$

Let us plot the graphs of $y=\sqrt{x}$, $y=\sqrt[3]{x}$, $y=\sqrt[4]{x}$, $y=\sqrt[5]{x}$. Since we do not have any idea about what they look like, we will construct a table of the values for certain x-values:

x	–4	–1	–0.25	0	0.25	1	4
$y=\sqrt{x}$	undefined	undefined	undefined	0	0.5	1	2
$y=\sqrt[3]{x}$	≈ -1.6	-1	≈ -0.6	0	≈ 0.6	1	≈ 1.6
$y=\sqrt[4]{x}$	undefined	undefined	undefined	0	≈ 0.7	1	≈ 1.4
$y=\sqrt[5]{x}$	≈ -1.3	-1	≈ -0.8	0	≈ 0.8	1	≈ 1.3

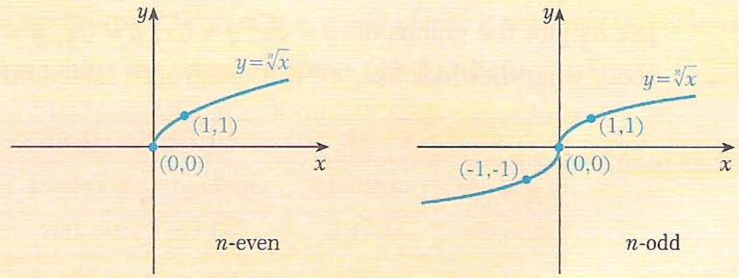
Using these data graphs will be as shown below:



Investigating the graphs above we can derive the following results:

Properties of the function $y=\sqrt[n]{x}$	
n even	n odd
The domain is $[0, \infty)$.	The domain is $\mathbb{R}$.
The range is $[0, \infty)$.	The range is $\mathbb{R}$.
The function is neither even nor odd.	The function is odd.
The graph passes through $(0, 0)$, $(1, 1)$.	The graph passes through $(-1, -1)$, $(0, 0)$, $(1, 1)$.
For $\|x\| > 1$; the bigger n is, the closer are the arms of the graph to the x-axis.	
For $\|x\| < 1$; the bigger n is, the closer are the arms of the graph to the y-axis.	

GRAPH OF $y = \sqrt[n]{x}$



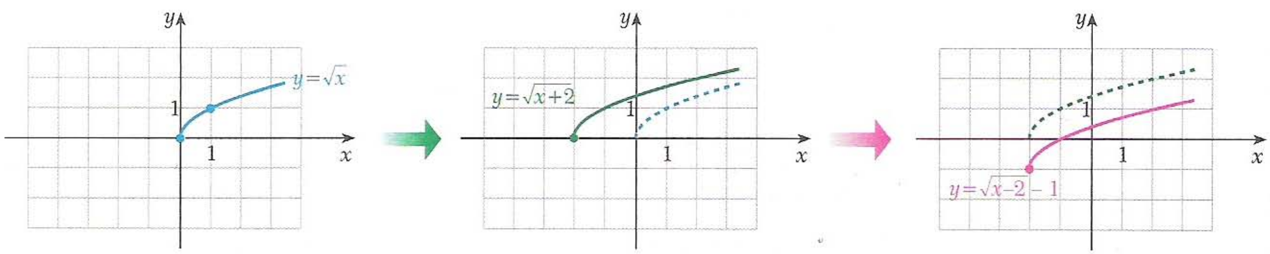
The function $y = \sqrt{x}$ is called the **square root function**. The point (0, 0) on the graph of the square root function is called the **end point**. The equation $y = a \cdot \sqrt{x-h} + k$ gives the end point form of the square root function where (h, k) is the end point.

With the help of the above scheme and transformation techniques we have learned so far we are now able to plot a rough graph of any function of the type $y = a \cdot \sqrt[n]{bx+c} + d$, where a, b, c, $d \in \mathbb{R}$.

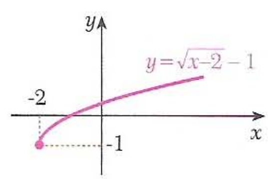
Example 93 Plot the graph of $y = \sqrt{x+2} - 1$.

Solution 1 This graph is a transformed form of $y = \sqrt{x}$. To obtain it we apply the following procedure:

1. Plot $y = \sqrt{x}$ graphing the common points (0, 0), (1, 1).
2. Plot $y = \sqrt{x+2}$ by horizontal shift (subtract 2 from each x-coordinate).
3. Plot $y = \sqrt{x+2} - 1$ by vertical shift (subtract 1 from each y-coordinate).



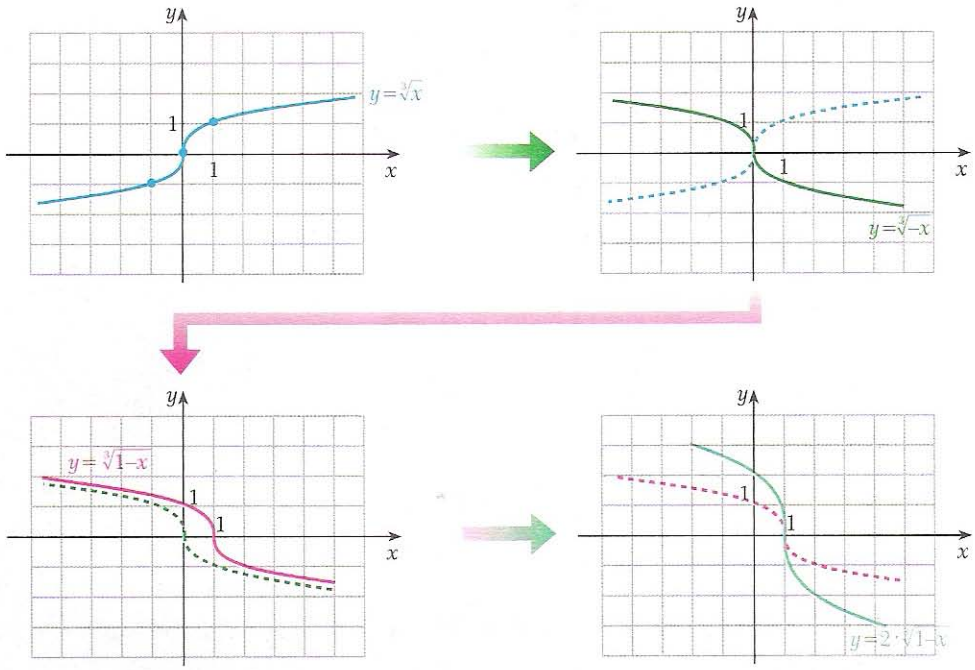
Solution 2 Clearly, $y = \sqrt{x+2} - 1$ is a square root function with the end point (–2, –1) and the domain [–2, ∞). So we have the following graph:



Example 94 Plot the graph of $y = 2 \cdot \sqrt[3]{1-x}$.

Solution This graph is a transformed form of $y = \sqrt[3]{x}$. To obtain it we apply the following procedure:

1. Plot $y = \sqrt[3]{x}$ graphing the common points $(-1, -1)$, $(0, 0)$, $(1, 1)$.
2. Plot $y = \sqrt[3]{-x}$ by horizontal reflection (symmetry with respect to the y-axis).
3. Plot $y = \sqrt[3]{-(x-1)} = \sqrt[3]{1-x}$ by horizontal shift (add 1 to each x-coordinate).
4. Plot $y = 2 \cdot \sqrt[3]{1-x}$ by vertical stretch (multiply each y-coordinate by 2).

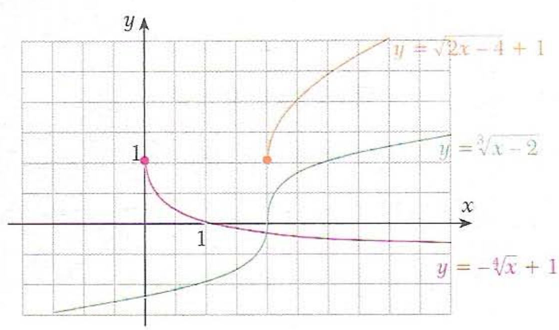


Check Yourself 21

Plot the graph of the following functions:

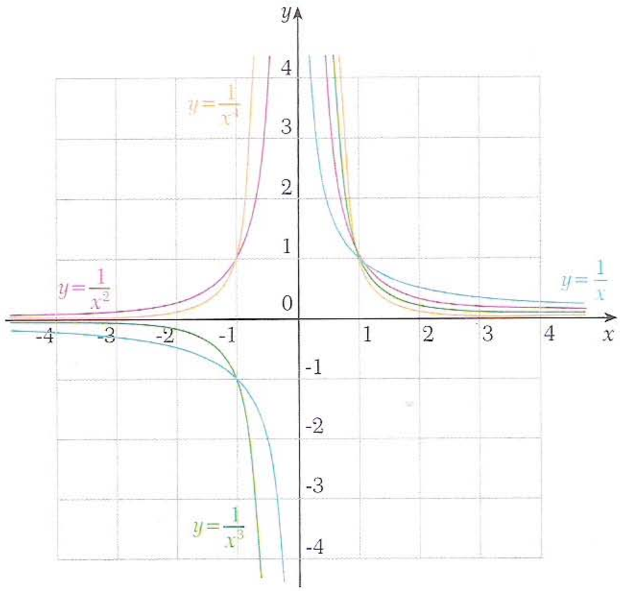
1. $y = -\sqrt[4]{x} + 1$ 2. $y = \sqrt[3]{x-2}$ 3. $y = \sqrt{2x-4} + 1$

Answers



3. Functions of the Form $y = \frac{1}{x^n}$, $n \in \mathbb{N}$

Let us plot the graphs of $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$, $y = \frac{1}{x^4}$. Since we do not have any idea about what they look like, we will construct a table of values for certain x-values. Using the data from the below table, the graphs will be constructed as demonstrated below on the left:

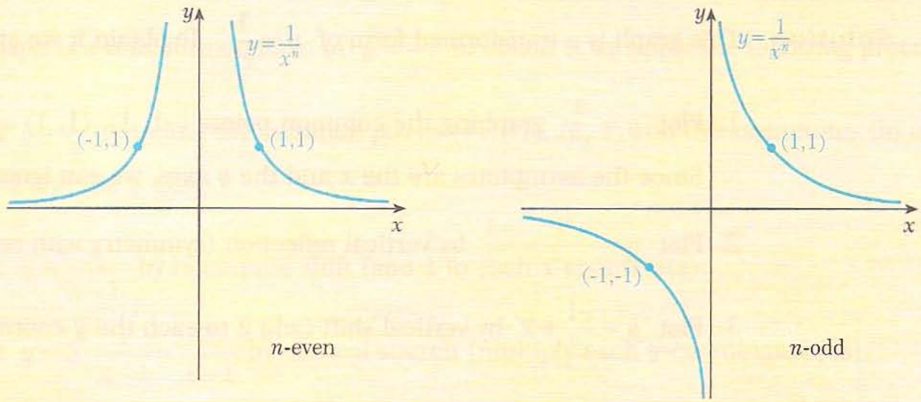


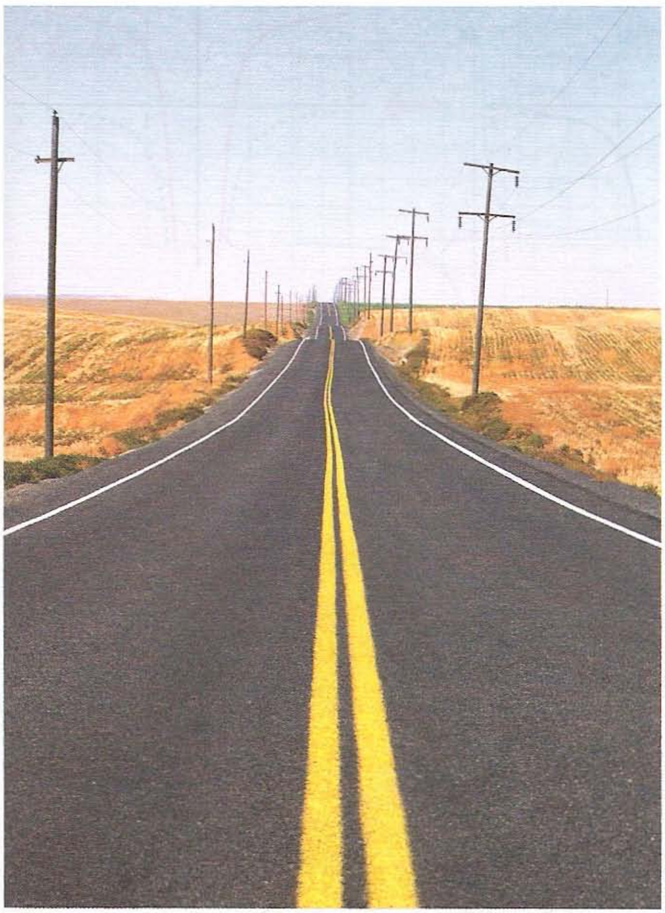
x	$y=\frac{1}{x}$	$y=\frac{1}{x^2}$	$y=\frac{1}{x^3}$	$y=\frac{1}{x^4}$
–2	–1/2	1/4	–1/8	1/16
–1	–1	1	–1	1
–1/2	–2	4	–8	16
0	undefined	undefined	undefined	undefined
1/2	2	4	8	16
1	1	1	1	1
2	1/2	1/4	1/8	1/16

Investigating the graphs above we can derive the following results:

Properties of the function $y = \frac{1}{x^n}$	
n even	n odd
The domain is $\mathbb{R} \setminus \{0\}$.	The domain is $\mathbb{R} \setminus \{0\}$.
The range is $\mathbb{R}^+$.	The range is $\mathbb{R} \setminus \{0\}$.
The function is even.	The function is odd.
The graph passes through (–1, 1), (1, 1).	The graph passes through (–1, –1), (1, 1).
For $\lvert x \rvert > 1$; the bigger n is, the closer are the arms of the graph to the x-axis.	
For $\lvert x \rvert < 1$; the smaller n is, the closer are the arms of the graph to the y-axis.	

GRAPH OF $y=\frac{1}{x^n}$, $n \in N$





The graphs seem to touch the asymptotes as both sides of the road seem to touch each other after some distance. But this never happens!

The graph of $y=\frac{1}{x}$ is called a **hyperbola**. The graph of $y=\frac{1}{x^2}$ is called a **truncus**.

All functions of the form $y=\frac{1}{x^n}$ show asymptotic behaviour. That is, as x becomes very large, the graph approaches the x-axis, but never touches it. As x becomes very small (approaches 0), the graph approaches the y-axis, but never touches it. So the line $x = 0$ (the y-axis) is a **vertical asymptote** and the line $y = 0$ (the x-axis) is the **horizontal asymptote**. The equation $y=\frac{a}{x-h}+k$ gives the asymptote form of the hyperbola where $x = h$ is the vertical asymptote and $y = k$ is the horizontal asymptote.

With the help of the above scheme and transformation techniques we have learned so far we are now able to plot the rough graph of any function of the type $y=\frac{a}{(bx+c)^n}+d$, where $a, b, c, d \in \mathbb{R}$ and $n \in \mathbb{N}$.

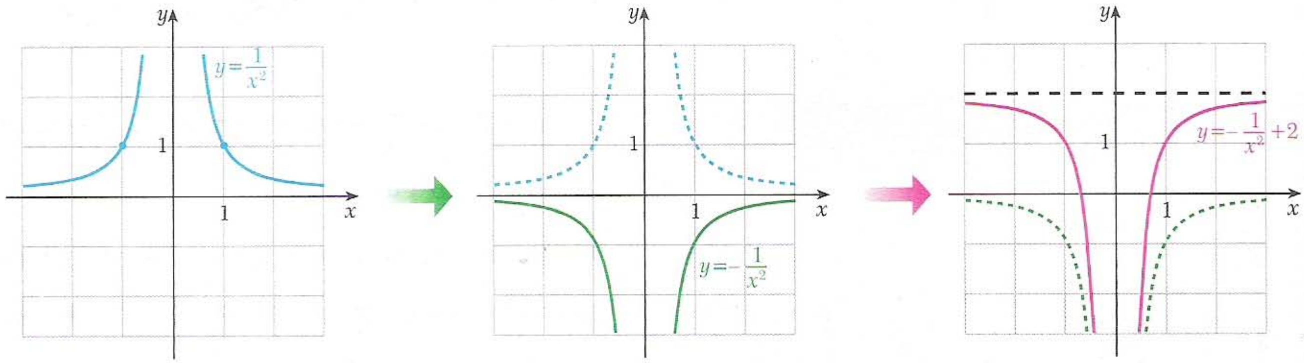
Example 95 Plot the graph of $y = \frac{-1}{x^2} + 2$.

Solution This graph is a transformed form of $y = \frac{1}{x^2}$. To obtain it we apply the following procedure:

1. Plot $y = \frac{1}{x^2}$ graphing the common points (–1, 1), (1, 1) and asymptotes $x = 0$, $y = 0$. Since the asymptotes are the x and the y axes, we can ignore an extra plot of them.

2. Plot $y = -\frac{1}{x^2} = \frac{-1}{x^2}$ by vertical reflection (symmetry with respect to the x-axis).

3. Plot $y = \frac{-1}{x^2} + 2$ by vertical shift (add 2 to each the y-coordinate).



Example 96 Plot the graph of $y=\dfrac{3}{x-1}+2$.

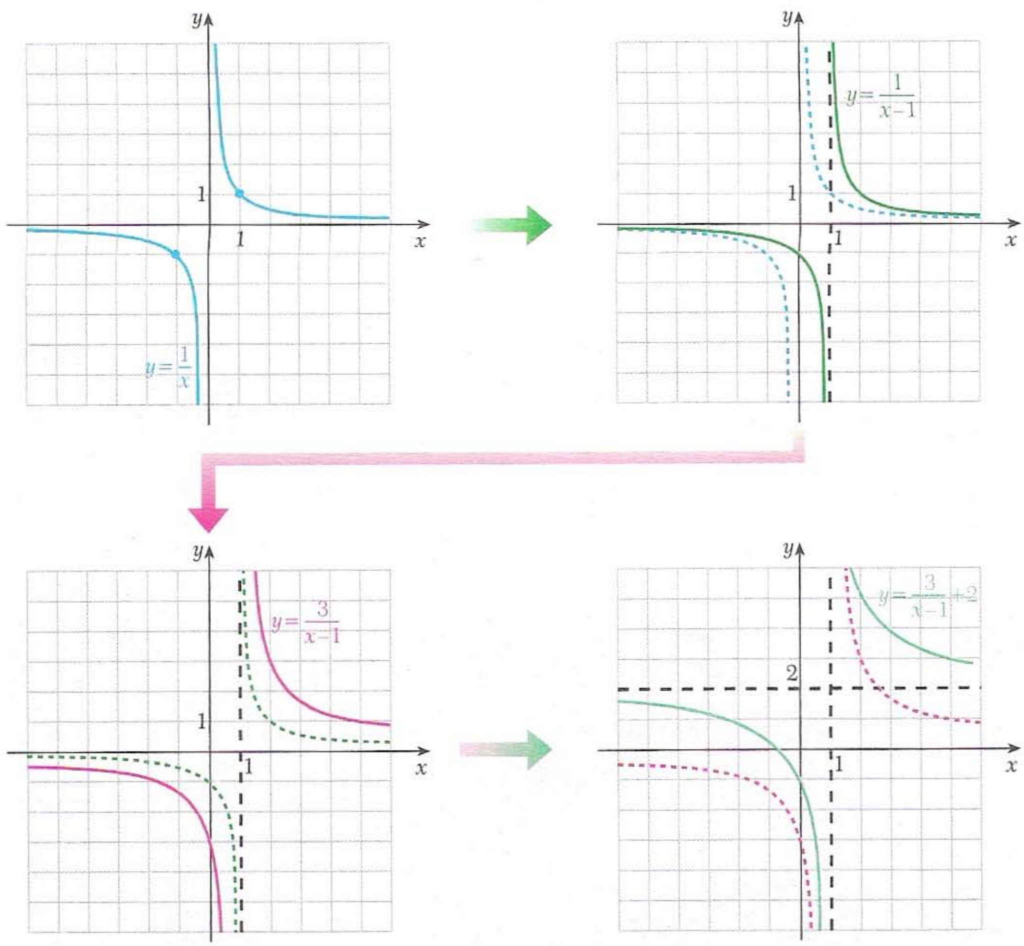
Solution This graph is a transformed form of $y=\dfrac{1}{x}$. To obtain it we apply the following procedure:

1. Plot $y=\dfrac{1}{x}$ graphing the common points (–1, –1), (1, 1) and the asymptotes (in our case the x and the y axes).

2. Plot $y=\dfrac{1}{x-1}$ by horizontal shift (add 1 to each x-coordinate).

3. Plot $y=3\cdot\dfrac{1}{x-1}=\dfrac{3}{x-1}$ by vertical stretch (multiply each y-coordinate by 3).

4. Plot $y=\dfrac{3}{x-1}+2$ by vertical shift (add 2 to each y-coordinate).



Example 97 Plot the graph of $y=\dfrac{2x-1}{x+3}$.

Solution 1 This function is not similar to any of the forms we have learned up to now. As demonstrated on the right we make a polynomial division of numerator by the denominator.

$$\begin{array}{r|l} 2x-1 & x+3 \\ \cline{2-2} \underline{-\;2x+6} & 2 \\ -7 & \end{array}$$

So, $y=\dfrac{2x-1}{x+3}=\dfrac{-7}{x+3}+2$ is a transformed form of $y=\dfrac{1}{x}$.

To obtain it we apply the following procedure:

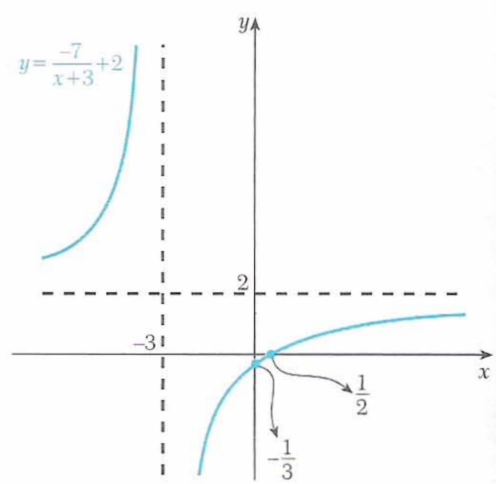
1. Plot $y=\dfrac{1}{x}$ graphing the common points $(-1, -1)$, $(1, 1)$.
2. Plot $y=\dfrac{1}{x+3}$ by horizontal shift (subtract 3 from each x-coordinate).
3. Plot $y=-7\cdot\dfrac{1}{x+3}=\dfrac{-7}{x+3}$ by vertical stretch (multiply each y-coordinate by -7).
4. Plot $y=\dfrac{-7}{x+3}+2$ by vertical shift (add 2 to each y-coordinate).

Plotting the graph with the help of the above transformations is left to the student. Below we have a faster way to plot the graph.

Solution 2 After finding that $y=\dfrac{2x-1}{x+3}=\dfrac{-7}{x+3}+2$, it is clear that we have a hyperbola with the asymptotes $x=-3$ and $y=2$. To have a more accurate graph let us choose a few points on the graph additionally. Usually, it is a good idea to choose intercepts:

If $x=0$, then $y=-\dfrac{1}{3}$.

If $y=0$, then $x=\dfrac{1}{2}$.

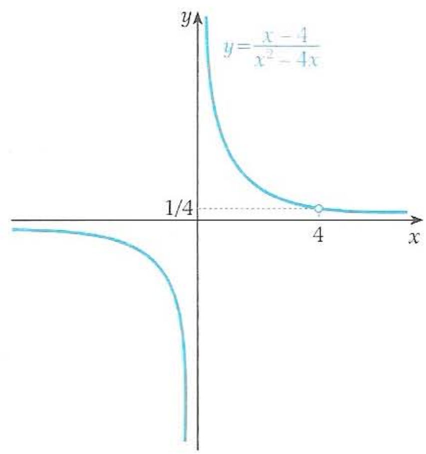


Note

A hyperbola of the form $y = \dfrac{ax+b}{cx+d}$ has $x = -\dfrac{d}{c}$ as the vertical asymptote and $y = \dfrac{a}{c}$ as the horizontal asymptote. Why?

Example 98 Plot the graph of $y = \dfrac{x-4}{x^2-4x}$.

Solution This function is also not a familiar one. But it is easy to see that $\dfrac{x-4}{x^2-4x} = \dfrac{1}{x}$. Here, it will be a critical mistake if we say that the graph of $y = \dfrac{x-4}{x^2-4x}$ is the same as the graph of $y = \dfrac{1}{x}$, since these functions are not equal. To understand this fact it is enough to note that the domain of $y = \dfrac{x-4}{x^2-4x}$ is $\mathbb{R} \setminus \{0, 4\}$, and the domain of $y = \dfrac{1}{x}$ is $\mathbb{R} \setminus \{0\}$. So if we want to get the correct picture we should plot $y = \dfrac{1}{x}$ with the domain $\mathbb{R} \setminus \{0, 4\}$ as on the right:



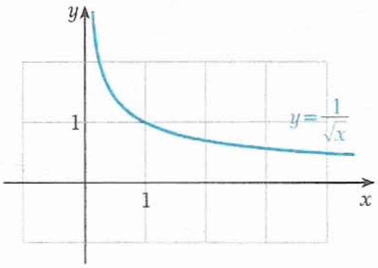
Example 99 Plot the graph of $y=\dfrac{1}{\sqrt{1-2x}}$.

Solution This function is a "mixture" of the square root function and the hyperbola. We can understand that it is a transformed form of $y=\dfrac{1}{\sqrt{x}}$, but we do not know the general shape of that graph.

After realizing that $y=\dfrac{1}{\sqrt{x}}$ is defined for $x>0$, let us construct a table of values for certain x-values:

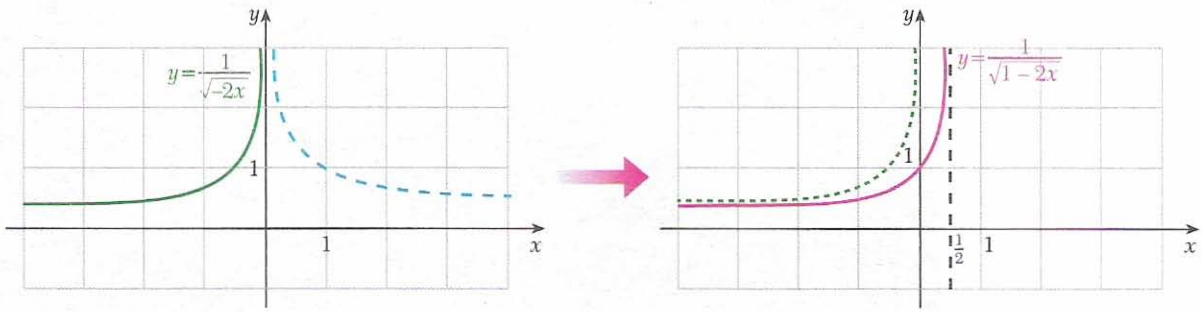
x	1/4	1	4
$y=\dfrac{1}{\sqrt{x}}$	2	1	1/2

Note that as x gets smaller y gets bigger and as x gets bigger y gets smaller provided that x and y never reaches 0. That is, $x=0$ and $y=0$ are asymptotes. So the graph will look like below:



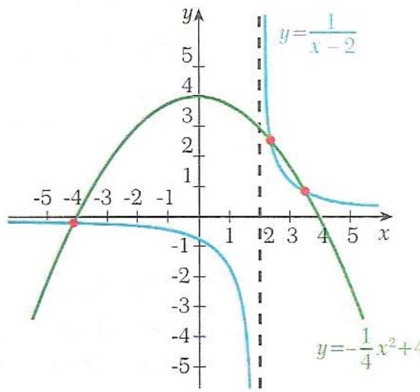
Now by the transformation we are ready to plot $y=\dfrac{1}{\sqrt{1-2x}}$ as below:

1. Plot $y=\dfrac{1}{\sqrt{x}}$.
2. Plot $y=\dfrac{1}{\sqrt{-2x}}$ by horizontal shrink with reflection (divide each x-value by –2).
3. Plot $y=\dfrac{1}{\sqrt{-2(x-\frac{1}{2})}}=\dfrac{1}{\sqrt{1-2x}}$ by horizontal shift (add $\dfrac{1}{2}$ to each x-coordinate).



Example 100 How many solutions does the equation $-\frac{1}{4}x^2+4=\frac{1}{x-2}$ have?

Solution If we multiply both sides of the equation by $4(x-2)\neq 0$, we will have $-x^3+2x^2+16x-32=4$ which we cannot solve by the techniques we know. However, realizing that we are not asked to solve the equation but to find the number of solutions, graphs may be an efficient tool. If we let the left-hand side of the equation $-\frac{1}{4}x^2+4=\frac{1}{x-2}$ be $f(x)=-\frac{1}{4}x^2+4$ and the right-hand side be $g(x)=\frac{1}{x-2}$, we will have the equation $f(x)=g(x)$.



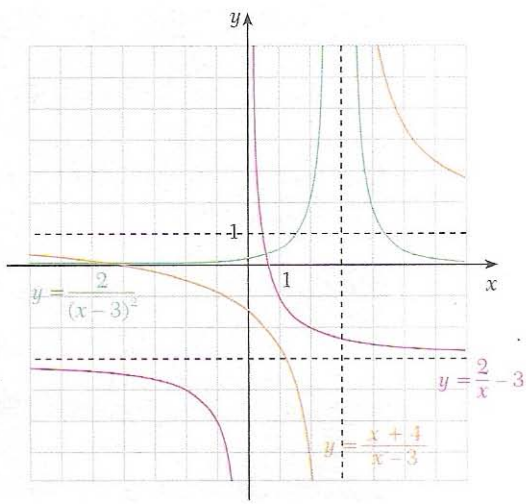
So the question of finding the number of solutions of the equation $-\frac{1}{4}x^2+4=\frac{1}{x-2}$ becomes a question of finding the number of the intersection points of the graphs of f and g. Above on the right we see the graphs of f and g. Since the graphs intersect at 3 points, the equation has 3 solutions.

Check Yourself 22

Plot the graph of the following functions:

1. $y=\frac{2}{x}-3$ 2. $y=\frac{2}{(x-3)^2}$ 3. $y=\frac{x+4}{x-3}$

Answers



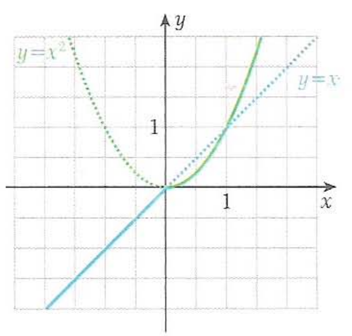
C. (OPTIONAL) GRAPHS OF SPECIAL FUNCTIONS

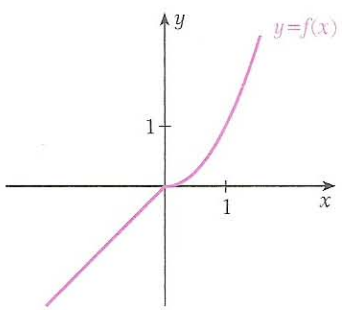
1. Piecewise Defined Functions

The functions graphed so far have been defined over their domains by a single rule. Some functions, however are defined by applying different rules at different parts of their domains. These kind of functions are called **piecewise defined functions**.

Consider the function $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$.

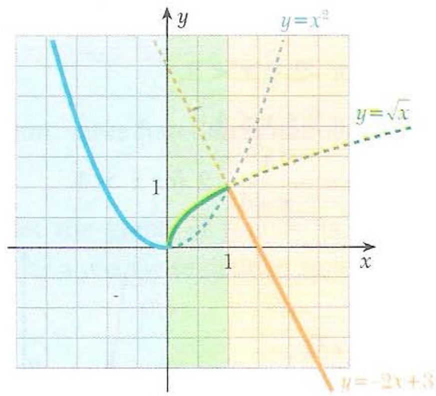
If we want to evaluate $f(3)$, we should consider the rule $f(x) = x^2$ since $3 \geq 0$, and if we want to evaluate $f(-5)$, we should consider the rule $f(x) = x$ since $-5 < 0$. In fact, for all $x \geq 0$ we have $f(x) = x^2$ and for all $x < 0$ we have $f(x) = x$. The same algorithm is applied to plotting the graph of that function. That is, the graph of this piecewise defined function will look like the graph of $y = x^2$ for $x \geq 0$ and $y = x$ for $x < 0$. So first let us plot both $y = x^2$ and $y = x$ on the same plane, then highlight the parts of the graphs that support the given condition. The final graph is below on the right:

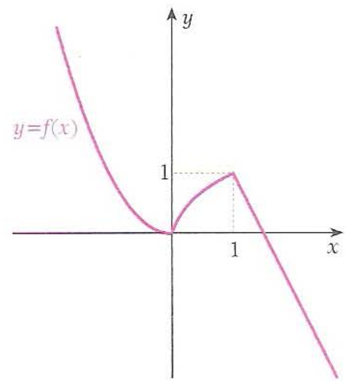




Example 101 Plot the graph of $f(x)=\begin{cases} x^2 & \text{if } x<0 \\ \sqrt{x} & \text{if } 0 \le x \le 1 \\ -2x+3 & \text{if } x>1 \end{cases}$.

Solution We plot the graph of parabola $y = x^2$ for $x < 0$, the square root function $y = \sqrt{x}$ for $0 \le x \le 1$ and the line $y = -2x + 3$ for $x > 1$. Taking the "necessary" parts of those graphs into consideration we get our final graph:

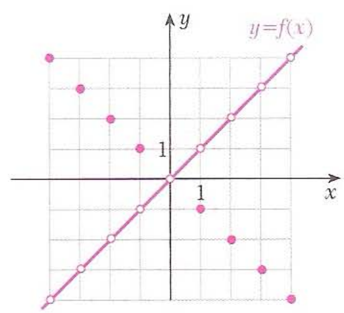




Note that in the above example graphs of each rule are connected to each other. So we can plot the complete graph without raising our hand. Later we will name these kind of functions as **continuous** functions.

Example 102 Plot the graph of $f(x)=\begin{cases} x & \text{if } x \notin \mathbb{Z} \\ -x & \text{if } x \in \mathbb{Z} \end{cases}$.

Solution For any non-integer x-value, graph will belong to the line $y = x$. For any integer x-value, graph will belong to the line $y = -x$. The graph below demonstrates these rules:

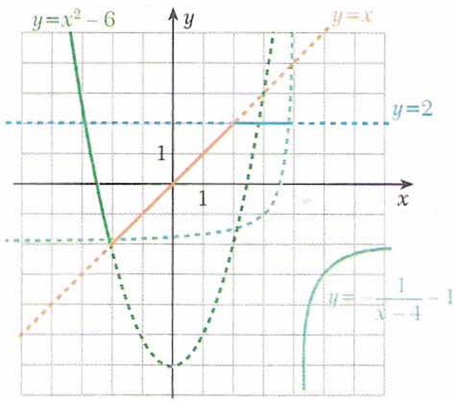


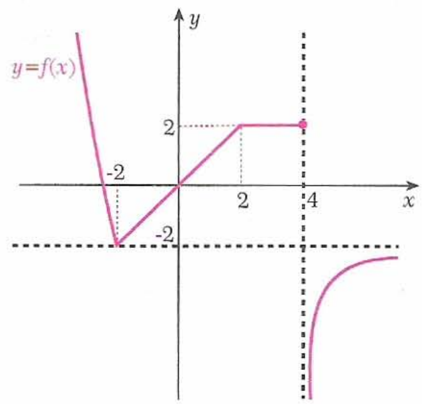
Note that in the above example graphs of each rule are not connected to each other. So we cannot plot the graph without raising our hand. Later we will name these kind of functions as **discontinuous** functions.

Example 103 Plot the graph of $f(x)=\begin{cases} x^2-6 & \text{if } x\le -2 \\ x & \text{if } -2<x<2 \\ 2 & \text{if } 2\le x\le 4 \\ -\dfrac{1}{x-4}-1 & \text{if } x>4 \end{cases}$.

Solution We should plot:

1. the graph of the parabola $y = x^2 - 6$ for $x \le -2$,
2. the graph of the line $y = x$ for $-2 < x < 2$,
3. the graph of the line $y = 2$ for $2 \le x \le 4$,
4. the graph of the hyperbola $y=-\dfrac{1}{x-4}-2$ for $x > 4$.





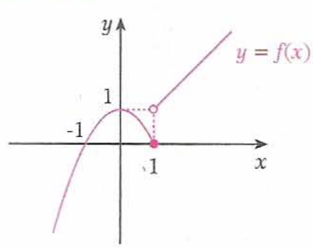
Check Yourself 23

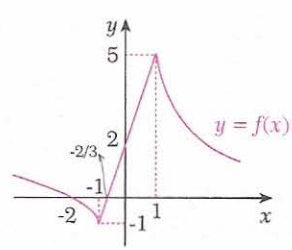
Plot the graph of the following functions:

1. $f(x)=\begin{cases} x & \text{if } x>1 \\ 1-x^2 & \text{if } x\le 1 \end{cases}$

2. $f(x)=\begin{cases} \sqrt{-x-1}-1 & \text{if } x\le -1 \\ 3x+2 & \text{if } -1<x<1 \\ \dfrac{5}{x} & \text{if } x\ge 1 \end{cases}$

Answers

1. 

2. 

2. Absolute Value Function

Recall that the absolute value of a number is equal to itself if the number is non-negative and is equal to the negative of itself if the number is negative. Symbolically, we have

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}.$$

The same definition holds for the absolute value function:

Definition

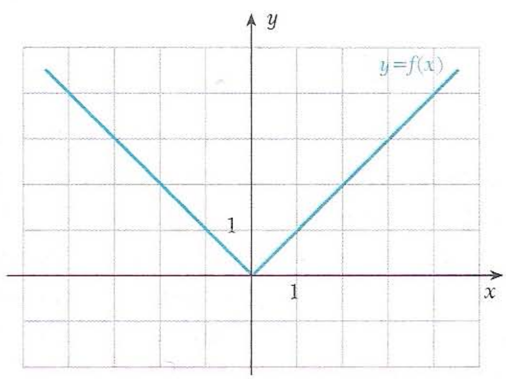
absolute value function

The **absolute value function** is defined as $|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$.

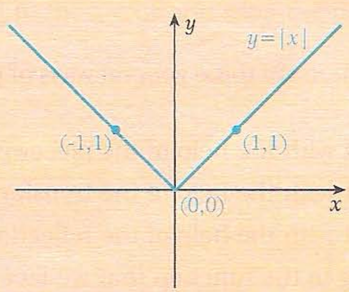
Let us plot the graph of $f(x) = |x|$. By definition, the absolute value function can be expressed as a piecewise defined function. So we have

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Plotting the graph of this piecewise defined function we will have the graph of $f(x) = |x|$:



THE GRAPH OF y = |x|



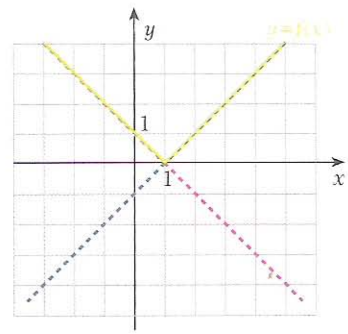
Example 104 Plot the graph of $f(x) = |x - 1|$.

Solution 1 Using the definition of the absolute value let us write the function $f(x) = |x - 1|$ as a piecewise defined function. In order to do that we should consider the cases that change the sign of the expression inside the absolute value:

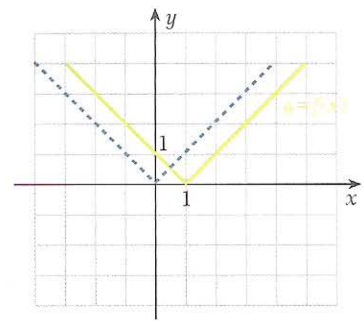
$$f(x) = \begin{cases} x-1 & \text{if } x-1 \geq 0 \\ -(x-1) & \text{if } x-1 < 0 \end{cases}$$

Simplifying the piecewise formula we have:

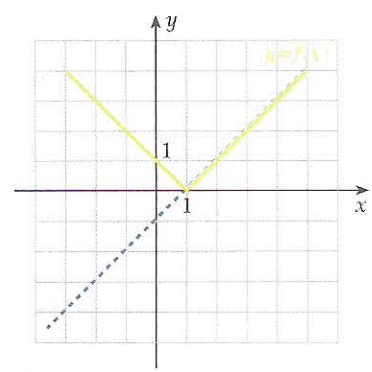
$$f(x) = \begin{cases} x-1 & \text{if } x \geq 1 \\ -x+1 & \text{if } x < 1 \end{cases}$$



Solution 2 Shifting the graph of $y = |x|$ to the right 1 unit we get the graph on the right:



Solution 3 The meaning of the absolute value is to keep the positive values as they are and to take the "negative" of negative values to make them non-negative. So the graph of $f(x) = |x - 1|$ can be obtained by plotting $y = x - 1$ and then reflecting the negative valued part in the x-axis so that it will be non-negative:



In fact, there are three general ways of plotting the graph of a function with an absolute value. They are

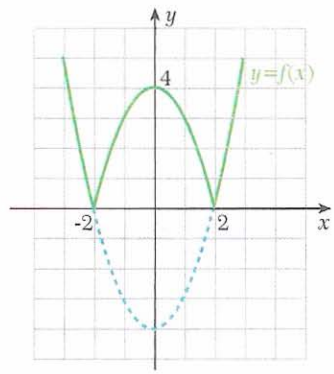
1. to plot with the help of the piecewise definition.
2. to plot with the help of the transformation.
3. to plot with the help of the reflection.

According to the function that we face one or more of these ways may be applicable.

Example 105 Plot the graph of $f(x) = |x^2 - 4|$.

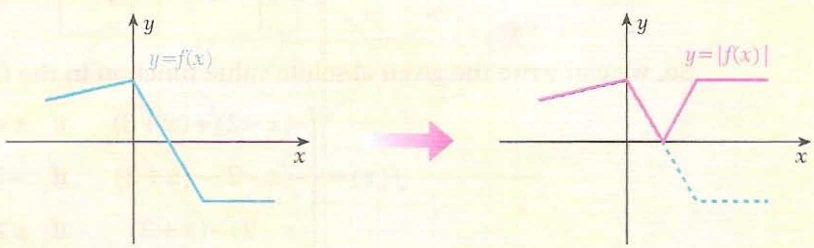
Solution Clearly, it will be simple to plot $y = x^2 - 4$ and reflect its negative valued parts in the x-axis:

Note that, in this example transformation would not work since we cannot obtain $f(x) = |x^2 - 4|$ by transformation from $f(x) = |x|$.



PLOTTING THE GRAPH OF $y = |f(x)|$

Given the graph of $f(x)$, to plot the graph of $|f(x)|$, take the symmetry of the negative part of the graph with respect to the x-axis and keep the positive part as previous.

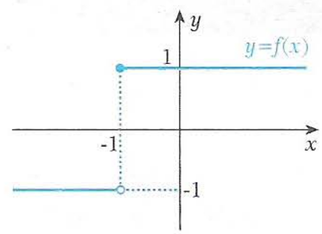


Example 106 Plot the graph of $f(x)=\dfrac{|x+1|}{x+1}$.

Solution This time a reflection will not work since the whole function is not inside an absolute value. Transformation is also not a good choice since we cannot obtain $f(x)=\dfrac{|x+1|}{x+1}$ by transformation of any elementary function we know. But defining this function piecewisely will always help. So we have

$$f(x)=\begin{cases}\dfrac{x+1}{x+1} & \text{if } x+1\geq 0\\[2ex] \dfrac{-(x+1)}{x+1} & \text{if } x+1<0\end{cases} \quad \text{which means } f(x)=\begin{cases}1 & \text{if } x\geq -1\\ -1 & \text{if } x<-1\end{cases}.$$

As demonstrated below, this graph is not difficult to plot:



EXAMPLE 107 Plot the graph of $f(x) = |x-2| - |x+3|$.

Solution Let us investigate the sign of each expression inside each absolute value with the help of the following table:

x	$-\infty$	-3		2	∞
$x-2$	$-$		$-$	0	$+$
$x+3$	$-$	0	$+$		$+$

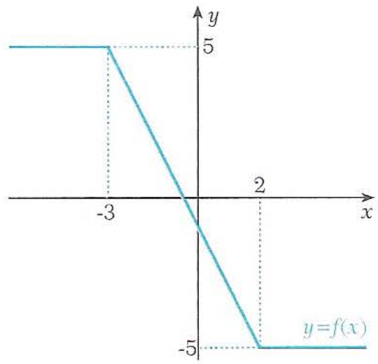
So, we can write the given absolute value function in the form of piecewise function:

$$f(x)=\begin{cases}-(x-2)+(x+3) & \text{if } x<-3\\ -(x-2)-(x+3) & \text{if } -3\leq x<2\\ (x-2)-(x+3) & \text{if } x\geq 2\end{cases}$$

Simplifying the piecewise defined function we have:

$$f(x)=\begin{cases}5 & \text{if } \ x<-3\\ -2x-1 & \text{if } \ -3\le x<2\\ -5 & \text{if } \ x\ge 2\end{cases}$$

The graph contains three parts and is as demonstrated below:



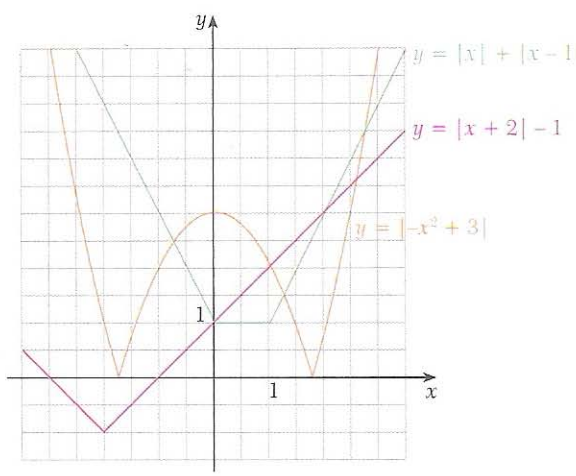
Above, note that for the first case we chose $x < -3$ and not $x \le -3$. This is just a notational detail. That is, we could also have chosen $x \le -3$. The important point here is that $x = -3$ should be included at least in one of the cases. Note that the second case has the condition $-3 \le x < 2$. Usually, as a rule for the absolute valued expressions to define the cases we choose the left border to be included and the right border to be excluded. The reason for that choice is hidden in the definition of the absolute value. Remember that $|a| = a$ if $a \ge 0$, and $|a| = -a$ if $a < 0$.

Check Yourself 24

Plot the graph of the following functions:

1. $f(x) = |x + 2| - 1$ 2. $f(x) = |-x^2 + 3|$ 3. $f(x) = |x| + |x - 1|$

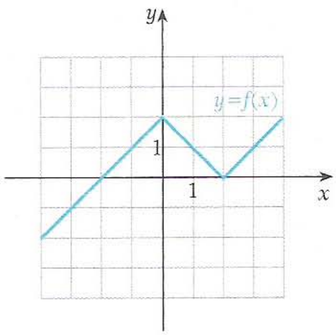
Answers



EXERCISES 4

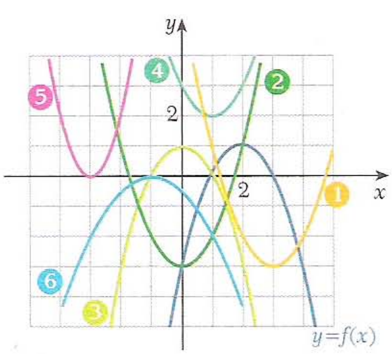
A. Transformation of Graphs

1. Given the graph of $f(x)$, plot the graphs of the following:

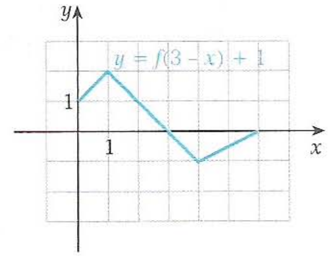


a. $3f(x)$
b. $f(x+2)$
c. $f(x)+2$
d. $f(2x)-1$
e. $f(2x-1)$
f. $-f(x-2)+3$
g. $\frac{1}{2}f(-x+1)-3$
h. $-2f(2x-6)$
i. $f(\frac{x}{3})+4$
j. $f(2x+4)-1$
k. $3f(-\frac{1}{2}x+2)+1$

2. Given the graph of $f(x)$, write the equation of each graph as a transformed form of $f(x)$.



3. Given the graph of $f(3-x)+1$, plot the graphs of the following functions:



a. $f(x)$
b. $-f(2x)$
c. $f(x+2)-1$

B. Graphs of Elementary Functions

4. With the help of the graph of $y = x^2$, plot the following graphs:

a. $y = (x-2)^2$
b. $y = x^2 - 1$
c. $y = (x-3)^2 - 1$
d. $y = (x+1)^2 + 4$

5. With the help of the graph of $\sqrt{x}$, plot the following graphs:

a. $y=\sqrt{x+3}$
b. $y=\sqrt{x}-1$
c. $y=\sqrt{x-4}+1$
d. $y=\sqrt{x+2}-2$

6. With the help of the graph of $y=\frac{1}{x}$, plot the following graphs:

a. $y=\frac{1}{x-2}$
b. $y=\frac{1}{x}+3$
c. $y=\frac{1}{x+4}+2$
d. $y=\frac{1}{x-3}-2$

7. Plot the graphs of the following functions:

a. $y = 3 - (x + 1)^2$ b. $y = 3(x - 2)^4 - 5$

c. $y = -x^3 + 3$ d. $y = (2x - 1)^3 - 1$

e. $y = -\dfrac{1}{3}(x-4)^6$ f. $y = 2 - \sqrt{x+1}$

g. $y = \sqrt{0.5x - 1}$ h. $y = 2\sqrt{2x+4}$

i. $y = \sqrt[3]{x+1} + 3$ j. $y = -\sqrt[4]{4-x} - 1$

k. $y = -\dfrac{2}{5-x}$ l. $y = \dfrac{-3}{x-4} + 2$

m. $y = \dfrac{-3}{x-2}$ n. $y = \dfrac{1}{2x+3} - 1$

o. $y = \dfrac{3x+1}{x-1}$ p. $y = \dfrac{-5x+3}{8-4x}$

q. $y = \dfrac{-4}{(3+2x)^2}$ r. $y = \dfrac{2}{(x+1)^2} - 1$

8. Plot the graphs of the following functions:

a. $y = \dfrac{1}{\sqrt{x-3}} - 2$

b. $y = \dfrac{x^4 - 2x^3}{x^2 - 2x}$

c. $y = x^3 - 6x^2 + 12x - 9$

d. $y = \dfrac{2x^2 + x - 1}{x^2 - 2x - 3}$

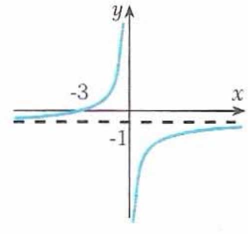
9. Find the equations of the following functions if they have one of the following forms:

$y = a(x-h)^2 + k$ $y = a(x-h)^3 + k$

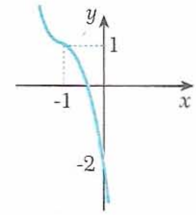
$y = \dfrac{a}{x-h} + k$ $y = \dfrac{a}{(x-h)^2} + k$

$y = a\sqrt{\mp x - h} + k$

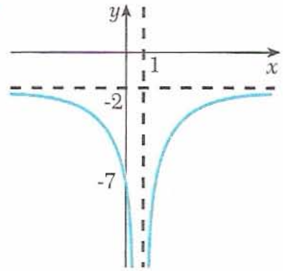
a.



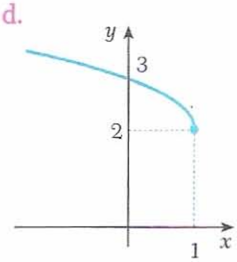
b.



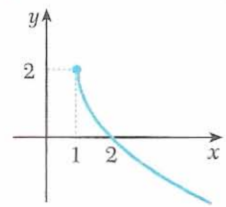
c.



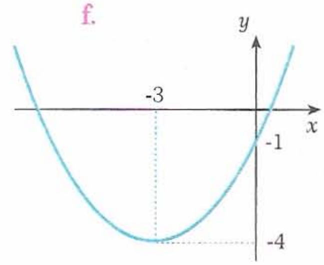
d.



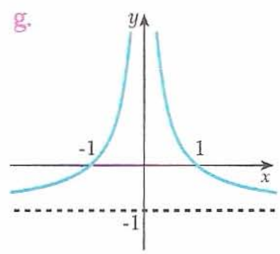
e.



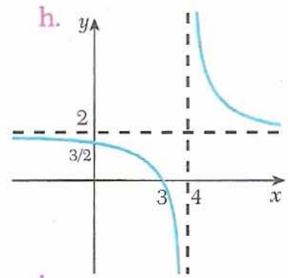
f.



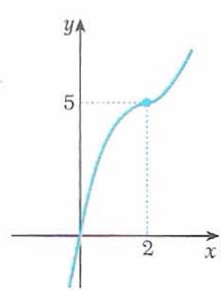
g.



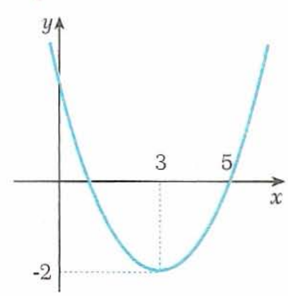
h.



i.



j.



C. Graphs of Special Functions

10. Plot the graphs of the following functions:

a. $y=\begin{cases} x+1 & \text{if } x>0 \\ 0.5 & \text{if } x=0 \\ x^2-1 & \text{if } x<0 \end{cases}$

b. $y=\begin{cases} 2 & \text{if } x<0 \\ 1-x^2 & \text{if } 0\le x<2 \\ x & \text{if } x\ge 2 \end{cases}$

c. $y=\begin{cases} \dfrac{12}{x} & \text{if } x<-2 \\ x^2+3x-4 & \text{if } -2\le x<1 \\ -\sqrt{x}+1 & \text{if } x\ge 1 \end{cases}$

d. $y=\begin{cases} x-1 & \text{if } x>3 \\ x^2-2x-3 & \text{if } 1<x\le 3 \\ x+4 & \text{if } x\le 1 \end{cases}$

11. With the help of the graph of $y = |x|$, plot the following graphs:

a. $y = |x + 2|$
b. $y = |x| - 2$
c. $y = |x - 3| + 1$
d. $y = |x + 4| - 3$

12. Plot the graphs of the following functions:

a. $y = |(x - 1)^4 - 1| - 1$

b. $y = |x^2 + 2x|$

c. $y=\left|\dfrac{1-2x}{x+2}\right|$

d. $y=\left|\sqrt{x+1}-2\right|$

e. $y = |4 + 2x| - |x - 1|$

f. $y=\dfrac{|x+1|}{x}$

g. $y = x + |x|$

13. Plot the graphs of the following functions:

a. $y = |x - 1| - 2|x| + 3|x + 2|$

b. $y=\dfrac{1}{|x-2|+|x|-3}$

c. $y = ||x + 1| - 2|x - 2|| - 3$

Mixed Problems

14. Given that $f(x)=\dfrac{4}{2+x}$ when $x > 0$, plot the complete graph if f is an odd function.

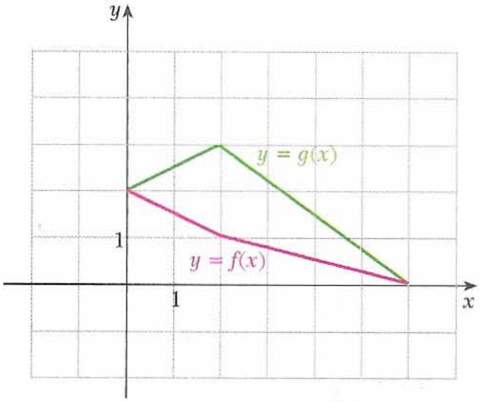
15. Given that $f(x)=2\cdot\sqrt[4]{1-x}-2$ when $x \le 0$, plot the complete graph if f is an even function.

16. Solve $x^3 - 7x + 6 < 0$ with the help of graphs.

17. Solve $\sqrt[4]{x-2}=10-x^2$ with the help of graphs.

18. Determine how many solutions the system $\begin{cases} xy=-4 \\ y-x^2=1 \end{cases}$ has with the help of graphs.

19. Given the graph of $f(x)$ and $g(x)$, plot the graphs of the following.



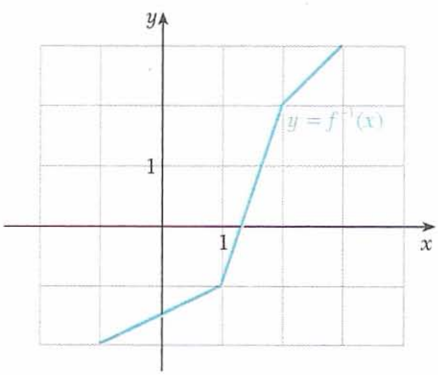
a. $f(x) + g(x)$

b. $f(x) - g(x)$

c. $2f\left(\frac{x}{2}\right) + g(-x+8)$

d. $|g(x) - 2f(x)|$

20. Given the graph of $f^{-1}(x)$, plot the graphs of the following.



a. $2f(x) - 3$

b. $f(x-1) + 2$

c. $-f(3x-6)$

21. Plot the graphs on an analytic plane supporting the following relations:

a. $x = y^2$

b. $|2x + y - 2| = 0$

c. $|x-1| + |y-3| = 1$

TESSELLATIONS

The human eye likes symmetric figures most of all. Therefore, symmetric patterns are used frequently as a design element by artists, architects, musicians, choreographers, etc. An interesting type of symmetry deals with tessellations which uses a combination of rotation, reflection and translation.

The word "tessellate" means to form or arrange small squares in a checkered or mosaic pattern. The word "tessellate" is derived from the Ionic version of the Greek word "tesseres", which in English means "four". Below you can see examples of tessellations:



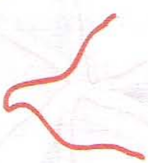

How to make your own:

With the help of an image editing software you can make your own tessellations and use it as a design element.

Method 1:

1. Draw a curve.

2. Copy the curve and then rotate it 60°.

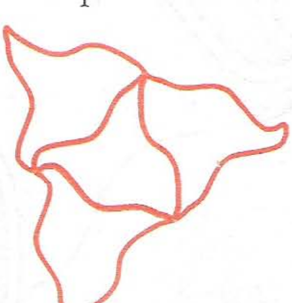


3. Draw a line joining the ends of the curves and mark its midpoint. This line will be removed later.

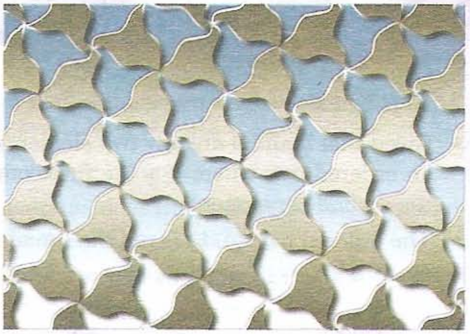
4. Draw another curve from one end of the previous curves to the midpoint of the line.

5. Copy and reflect the last curve horizontally and vertically, then translate it to fill the gap in the figure.

6. The final figure will always tessellate. Copy and paste the pieces together. You will need to rotate some of them with a multiple of 60° to fit.



If you use different colors in alternating pieces, they will show up better and not run into each other. Further artistic design is left to you!



Method 2:

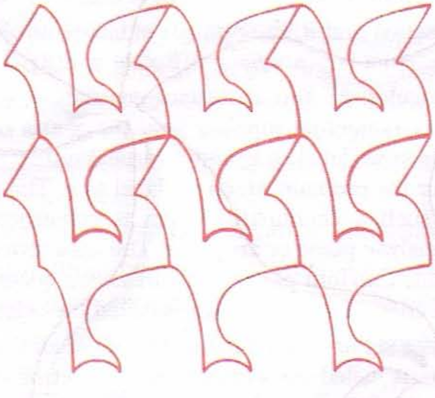
1. Draw a square.

2. Mark out a shape to be cut out on one side.

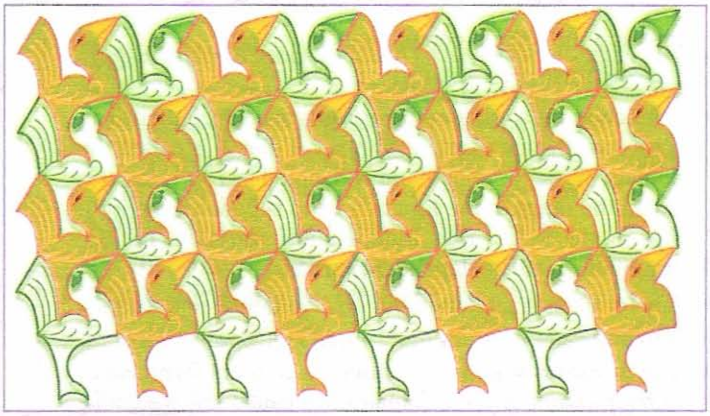
3. Cut out the shape and place it on the opposite side.

4. Repeat the same procedure for the remaining parallel sides using another shape.

5. Now copy and paste as much as you like.



A little art would help us to improve the final image.



Remember the transformation types that are used in plotting graphs of functions. Which transformation types did we use in the methods described above? What kind of shapes will tessellate? Can you find another tessellation method? Try to make your own tessellations.

CHAPTER SUMMARY

1. Relations

- Given that a and b are any two elements, (a, b) is called an **ordered pair** where a is the **first component** and b is the **second component**.
- The ordered pair (a, b) is not the same as (b, a).
- $(a, b) = (c, d)$ if, and only if, $a = c$ and $b = d$.
- Let A and B be two non-empty sets. Then the set of all ordered pairs whose first component is from A and whose second component is from B is called the **cartesian product** of A and B and is denoted by $A \times B$.
- Given two sets A and B, the number of elements of their cartesian product is $n(A \times B) = n(B \times A) = n(A) \cdot n(B)$.
- The **rectangular** or the **cartesian coordinate system** consists of a horizontal number line, the **x-axis** or the **abscissa** which we label as x, and a vertical number line, the **y-axis** or the **ordinate** which we label as y. The plane on which such a coordinate system is constructed is called an **analytic plane** or **xy-plane**. The axes divide the analytic plane into four parts which are called **quadrants**. The intersection point of the axes is called the **origin**.
- A **relation** is a set of ordered pairs. The set of all the first components is called the **domain** and the set of all the second components is called the **range** of the relation. A relation can be represented by a **list**, by a **table**, by a **map**, or by a **graph**.
- The inverse of a relation is obtained by interchanging the first and the second components.

2. Introduction to Functions

- A **function** f is a relation that assigns to each element x in set A exactly one element y in set B. Set A is called the **domain** of the function f and is denoted by $D(f)$. Set B is called the **range** of the function f and is denoted by $E(f)$. We name x as the **independent variable** or the **argument**, y as the **dependent variable**.
- Any function is a relation but any relation is not always a function.
- The functions which are defined by different formulae in different parts of their domains are called the **piecewise defined functions**.
- If the graph of a function crosses the x-axis, then the function has an **x-intercept**. The x-intercept(s) of a function are called the **zeros** or the **roots** of the function.
- If the graph of a function crosses the y-axis, then the function has a **y-intercept**. A function can have at most one y-intercept.
- A set of points in the coordinate plane is the graph of a function if, and only if, no vertical line crosses the graph at more than one point.
- To find the domain of a function we apply the following basic principles:
  1. The polynomial functions have any real number as their domain.
  2. The rational functions have any real number except the ones that make the denominator zero as their domain.
  3. The even degreed root functions (square root, fourth degree root, etc.) have any real number that makes the expression under the root non-negative as their domain.
  4. The odd degreed root functions (cubic root, fifth degree root, etc.) have any real number as their domain.
  5. If a function contains the combination of the above functions then the domain is found by taking the intersection of all conditions.
- Two functions $f(x)$ and $g(x)$ are equal if, and only if, $f(x) = g(x)$ and $D(f) = D(g)$.
- A function f is **even** if, for each x in its domain, $f(-x) = f(x)$. The graph of an even function is symmetric with respect to the y-axis. A function f is **odd** if, for each x in its domain, $f(-x) = -f(x)$. The graph of an odd function is symmetric with respect to the origin. Generally the functions that we face are neither even nor odd.
- A function f is **increasing** on an interval I if $f(x_1) < f(x_2)$ for any $x_1 < x_2$ in I. A function f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ for any $x_1 < x_2$ in I. A function f is **constant** on an interval I if $f(x_1) = f(x_2)$ for any $x_1 < x_2$ in I.
- We read the graph of a function the same way that we would read a book, from left to right. The behavior of the function is determined by the y-values, but the intervals are reported in terms of the x-values.

3. Operations on Functions

- $f(g(x))$ is defined as the **composition** of f and g. It is also denoted by $(f \circ g)(x)$, or simply $f \circ g$.
- $f \circ g$ means g is applied first, f is applied second and in general $f \circ g \neq f \circ g$.
- A function f is **one-to-one** if, for each $x_1 \neq x_2$ in its domain, $f(x_1) \neq f(x_2)$. The graph belongs to a one-to-one function if, and only if, no horizontal line crosses itself at more than one point.
- A function has an inverse if, and only if, it is one-to-one and onto.

- Let f be a one-to-one and onto function with a domain A and a range B. Then its inverse function f^{-1} has a domain B and a range A such that $f(x) = y \Leftrightarrow f^{-1}(y) = x$.
- $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$
- $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.
- To plot the graph of inverse of a function reflect the graph of the function with respect to the line $y = x$. To find the formula for the inverse function solve the equation $y = f(x)$ for x and interchange x and y.

4. Plotting Graphs of Functions

- Given $c \in \mathbb{R}$, after the transformation coordinates of each point, (x,y) on the graph of $y = f(x)$ change as follows:

Name	Function	Coordinates
Vertical shift	$y = f(x) + c$	$(x, y + c)$
Horizontal shift	$y = f(x + c)$	$(x - c, y)$
Vertical reflection	$y = -f(x)$	$(x, -y)$
Horizontal reflection	$y = f(-x)$	$(-x, y)$
Vertical stretch and shrink	$y = cf(x)$	(x, cy)
Horizontal stretch and shrink	$y = f(cx)$	$(x/c, y)$

- Given the graph of $f(x)$ and $a, b, c, d \in \mathbb{R}$, to plot the graph of $a \cdot f(bx + c) + d$ we apply the following transformation order

Step	Action	Explanation
1	Plot $f(bx)$	Divide each x value by b
2	Plot $f(b(x+\frac{c}{b}))$	Subtract $\frac{c}{b}$ from each x value
3	Plot $af(bx + c)$	Multiply each y-value by a
4	Plot $af(bx + c) + d$	Add d to each y value

- The absolute value function is defined as

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

- In fact, there are three general ways of plotting a graph of a function with an absolute value. They are:
  1. to plot with the help of the piecewise definition.
  2. to plot with the help of the transformation.
  3. to plot with the help of the reflection.
- Given the graph of $f(x)$, to plot the graph of $|f(x)|$, take the symmetry of the negative part of the graph with respect to the x-axis and keep the positive part as previous.

Concept Check

- Where can we represent elements of the cartesian product of two real number sets? What about the cartesian product of three real number sets?
- In which ways can we represent a relation?
- How many quadrants does the analytic plane have? How do we name the x and the y-axes?
- Postage cost is a function of weight, age is a function of time. Give three other examples of functions from everyday life.
- Is it true that every relation is a function? Is it true that every function is a relation? If not, give an example.
- Is it possible that $f(g(x)) = g(f(x))$? If so, give an example.
- What is the domain and the range of a function? Give an example for a function whose domain is any real number but whose range is not. Can you give an example for a function whose range is any real number but whose domain is not?
- How can we find the domain of a function?
- State the vertical line test. When do we use this test?
- State the horizontal line test. When do we use this test?
- When can we say that two functions are equal?
- Why can a function have at most one y-intercept?
- Can an even function be one-to-one? What about an odd function?
- Is it true that all functions have an inverse? If not, which functions have an inverse?
- Can an inverse of a function be a constant function? Why?
- Is it true that if a function is always increasing or decreasing in its domain then it is one-to-one?
- How can we find a formula for the inverse of a function? What about plotting a graph of the inverse?
- Give examples of symmetry from real life.
- What must be true about the integer n if the function $f(x) = x^n$ is an even function? If it is an odd function?
- What is the effect of $n \in \mathbb{N}$ in the graph of $y = x^n$? What about $\sqrt[n]{x}$ and $\frac{1}{x^n}$?
- What is asymptote? Which functions' graphs have asymptotes?
- In the graph of $a\,f(bx + c) + d$ which is obtained from the graph of $f(x)$, explain the effect of a, b, c, d.
- How can we plot the graph of a function with an absolute value?

1. If $(3k, 4) = (9, p + 1)$, find $p \cdot k$.

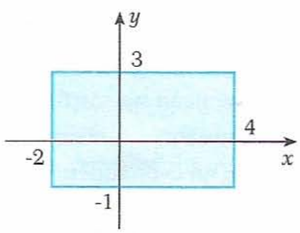
A) 6 B) 9 C) 10 D) 15 E) 18

2. Given that $A = \{a, b, c\}$ and $B = \{a, b\}$, which one of the following is not an element of $A \times B$?

A) (a, a) B) (c, b) C) (b, a)

D) (c, a) E) (a, c)

3. How can the following graph be expressed as a cartesian product?



A) $[-2, 4] \times [-1, 3]$ B) $[-1, 3] \times [-2, 4]$

C) $(-2, 4) \times (-1, 3)$ D) $(-1,3) \times (-2, 4)$

E) $[-2, 3] \times [-1, 4]$

4. Which one(s) of the following relations are functions?

I. $\{(1, 2), (0, 5), (1, 4)\}$
II. $\{(a, a), (b, a), (c, a)\}$
III. $\{(2, 4), (-3, 5), (3, 5)\}$

A) only I and II B) only II and III

C) I, II and III D) only II

E) only III

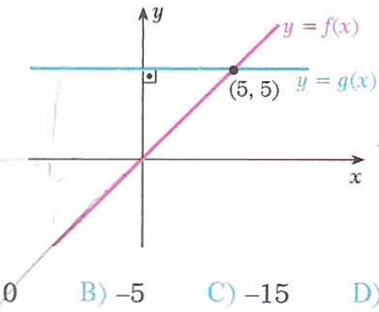
5. Given that $f(x) = \sqrt{x-4}$, find the domain and the range of the function.

A) $(4, \infty)$ and $(-\infty, \infty)$ B) $[4, \infty)$ and $(0, \infty)$

C) $(4, \infty)$ and $[0, \infty)$ D) $[4, \infty)$ and $[0, \infty)$

E) $(4, \infty)$ and $(0, \infty)$

6. Using the figure below, find $f(-10) + g(-10)$.



-10 +10

A) -10 B) -5 C) -15 D) 5 E) 0

7. If $f(x) = 2x - 1$ and $g(x) = x^2 - x + 3$, find $(g + f)(3)$.

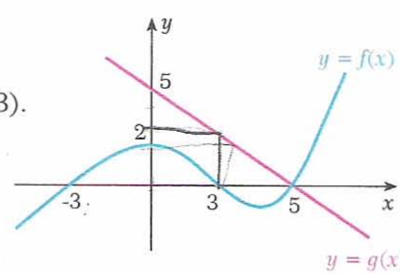
A) 8 B) 10 C) 12 D) 14 E) 16

8. If $f(x) = \sqrt{x+2}$ and $g(x) = x^4 + x^2 - 1$, find $(g \circ f)(3)$.

A) 29 B) 16 C) 13 D) 19 E) -17

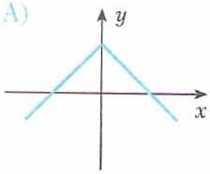
9. Using the figure, find $g(3) + (g \circ f)(3)$.

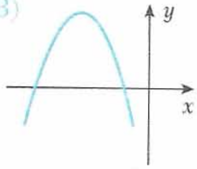
3 + 5

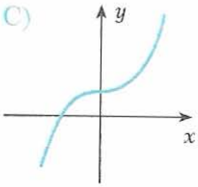


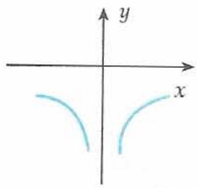
A) 3 B) 5 C) 7 D) 9 E) 12

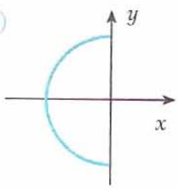
10. Of the graphs below which one represents a one-to-one function?

A)


B)


C)


D)


D)


11. If $f = \{(3, 4), (7, 9), (13, 13), (17, 20)\}$, find f^{-1}.

A) $\{(4, 3), (9, 7), (13, 13), (20, 17)\}$

B) $\{(3, 4), (7, 9), (13, 13), (17, 20)\}$

C) $\{(3, 4), (9, 7), (13, 13), (17, 20)\}$

D) $\{(4, 3), (9, 7), (13, 13), (17, 20)\}$

E) $\{(3, 4), (7, 9), (13, 13), (20, 17)\}$

12. If $f(x) = mx + n - 2$, find $f^{-1}(x)$.

A) $mx - n + 2$

B) $\dfrac{x+n-2}{m}$

C) $\dfrac{x+2-n}{m}$

D) $mx + 2$

E) $\dfrac{x-n}{m}$

13. If $f(x) = 7x + 5$, find $f^{-1}(3)$.

A) 26 B) 8 C) $\dfrac{1}{7}$ D) $-\dfrac{5}{7}$ E) $-\dfrac{2}{7}$

14. Given the graph of $y = f(x + a) - b$, which one of the following is correct?

A) Increase in a will result in the shift of the graph to the right.

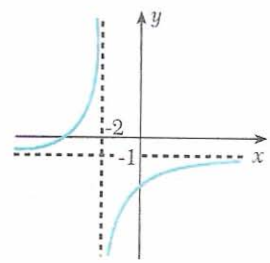
B) Decrease in b will result in the shift of the graph up.

C) Decrease in a will result in the shift of the graph down.

D) Increase in b will result in the shift of the graph up.

E) Decrease in a will result in the shift of the graph to the left.

15. Which one of the equations below is most likely related to the following graph?



A) $y = -\dfrac{2}{x-2} - 1$

B) $y = 1 - \dfrac{2}{x+2}$

C) $y = -\dfrac{2}{x+1} - 2$

D) $y = \dfrac{2}{x+2} - 1$

E) $y = \dfrac{-2}{x+2} - 1$

16. The equation of a parabola is given by $y = p - 2(x + 3)^2$. The increase in p will result in:

A) the graph being thinner.

B) the graph being wider.

C) the increase of the domain.

D) the increase of the range.

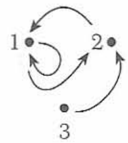
E) the graph being shifted to the left.

CHAPTER REVIEW TEST 2

1. If the point $A(4 - x,\ 7 - x)$ is on the second quadrant of the analytic plane, find the sum of the all possible integer values of x.

A) 9 B) 10 C) 11 D) 12 E) 13

2. Which one of the following is the representation of the following relation by the list method?



A) $\{(2, 2), (3, 2), (2, 1), (1, 1)\}$

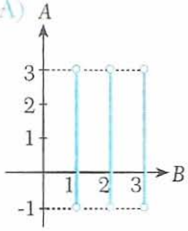
B) $\{(1, 1), (2, 2), (2, 3), (2, 1)\}$

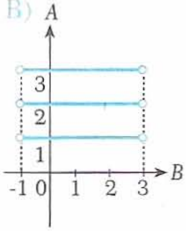
C) $\{(2, 3), (2, 1), (1, 1), (3, 3)\}$

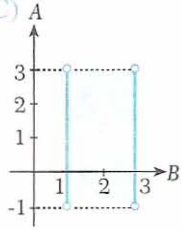
D) $\{(1, 2), (2, 2), (2, 3), (3, 3)\}$

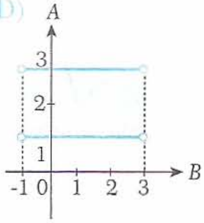
E) $\{(2, 1), (3, 2), (1, 2), (1, 1)\}$

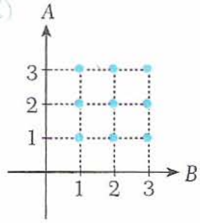
3. If A = {1, 2, 3} and B = (–1, 3), which one of the following represents $B \times A$?

A) 

B) 

C) 

D) 

E) 

4. Find the domain of $f(x) = \sqrt{x-2} + \dfrac{15}{x-10}$.

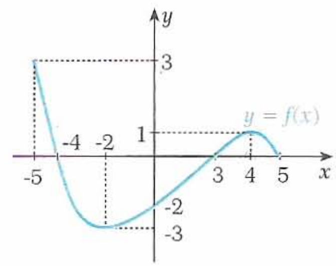
A) $[2, \infty)$ B) $[2, 10]$ C) $[2, 10) \cup (10, \infty)$

D) $\mathbb{R}\setminus\{10\}$ E) $\{2, 10\}$

5. If $f(x) = \dfrac{2x-5}{x-2}$, find $f(\frac{1}{2})$.

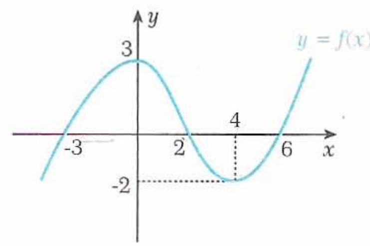
A) $\dfrac{3}{8}$ B) $\dfrac{3}{4}$ C) $\dfrac{8}{3}$ D) $\dfrac{4}{3}$ E) 6

6. Using the figure find $\dfrac{f(4)-f(0)}{f(-2)-f(3)}$.



A) $-\dfrac{2}{3}$ B) $-\dfrac{1}{3}$ C) -1 D) $\dfrac{2}{3}$ E) 1

7. Using the figure which one(s) of the following are correct?



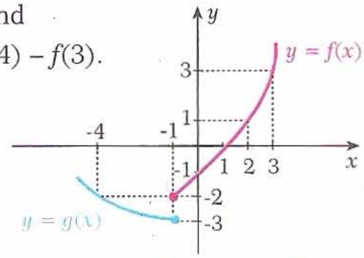
I. The function is decreasing on [0, 4].
II. –3, 2 and 4 are zeros of the function.
III. The function is neither even nor odd.
IV. $f(6) = 0$

A) only I and II B) only III and IV

C) all D) only I, IIand IV

E) only I, III and IV

8. Using the figure find
$(g \circ f^{-1})(-2) - g(-4) - f(3)$.



-3 + 2 - 3

A) 2 B) 1 C) –3 D) –4 E) –8

9. At which one of the following points is it possible for a function and its inverse to intersect?

A) (1, 2) B) (2, 1) C) (1, 1)
D) (–1, 0) E) (1, –1)

10. If $f(x) = x^2 + 3$ and $g(x) = 2x - 1$, find $(g \circ f)(x)$.

A) $4x^2 + 4x + 4$ B) $4x^2 + 4x + 1$
C) $2x^2 + 5$ D) $2x^2 + 7$
E) $2x^2 - 3$

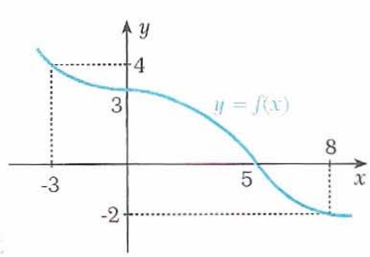
11. If $f(x) = \dfrac{3x-3}{x+2}$, find $f^{-1}(x)$.

A) $\dfrac{3x+3}{x-3}$ B) $\dfrac{2x+3}{-x+3}$ C) $\dfrac{2x-3}{x-3}$
D) $\dfrac{x+3}{2-x}$ E) $\dfrac{3x-3}{x+2}$

12. Find $f\left(f^{-1}\left(\dfrac{1}{2}\right)\right)$.

A) 2 B) 1 C) $\dfrac{1}{2}$ D) x E) –2

13. Using the figure find $\dfrac{f(5)+f^{-1}(0)}{f^{-1}(4)+f(8)}$.



A) 3 B) 2 C) –1 D) –2 E) –5

14. Given the graph of $y = f(x)$ to get the graph of $y = -f(x) + 3$ which procedure should be applied?

A) Reflect the graph with respect to the y-axis and shift 3 units to the left.

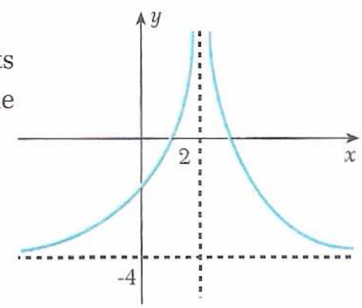
B) Reflect the graph with respect to the x-axis and shift 3 units up.

C) Reflect the graph with respect to the y-axis and shift 3 units down.

D) Reflect the graph with respect to the x-axis and shift 3 units down.

E) Reflect the graph with respect to the y-axis and shift 3 units up.

15. Which of the following statements is not correct for the following graph?



A) The vertical asymptote is $x = 2$.

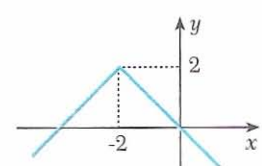
B) The horizontal asymptote is $y = -4$.

C) The range is $R \setminus \{-4\}$.

D) The domain is $R \setminus \{2\}$.

E) There are two x-intercepts.

16. Which one of the equations below is most likely related to the following graph?

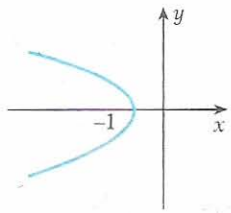


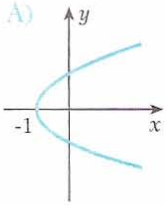
A) $y = |x + 2| + 2$ B) $y = 2 - |x + 2|$
C) $y = 2|2 - x| + 2$ D) $y = 2 - |2 - x|$
E) $y = |x + 2| - 2$

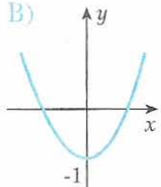
1. Given the relation β = {all (x, y) such that $2x + 3y = 7$ and $x, y \in \mathbb{Z}^+$}, which one of the following is an element of it?

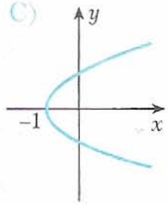
A) (2, 1) B) (1, 2) C) (1, 1) D) (2, 2) E) (0,2)

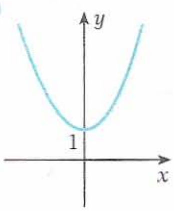
2. Which one of the graphs below represents the inverse of the following relation best?

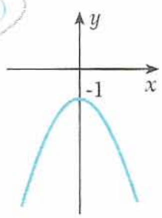


A)


B)


C)


D)


E)


3. The relations below are defined on $A \times B$ where $A = \{2, 3, 5\}$ and $B = \{4, 9, 36\}$. How many of them can be considered as a function?

{all (x, y) such that x divides y}

{all (x, y) such that $3x - y > 0$ }

{all (x, y) such that $y = x^2$}

{all (x, y) such that $x + y$ is prime}

{all (x, y) such that $x + y$ is odd}

A) 0 B) 1 C) 2 D) 3 E) 4

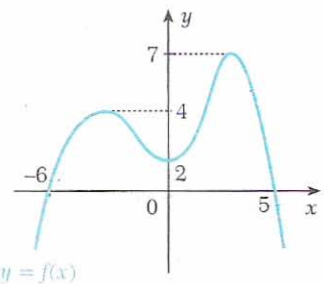
4. If $f(x + 1) + f(x) = 3x - 1$ and $f(5) = 3$, find $f(7)$

A) 10 B) 9 C) 8 D) 7 E) 6

5. If $\dfrac{f(5)+f^{-1}(0)}{f^{-1}(4)+f(8)}$. $f(x + 1) + f(x) = 3x - 1$ and $f(5) = 3$, find $f(7)$

A) 14 B) 15 C) 16 D) 17 E) 18

6. Using the figure on the right, state how many roots the equation $f(x) = 3$ has.



A) 5 B) 4 C) 3 D) 2 E) 1

7. If $f(x) + 2f(-x) = x + 4$, find $f(x)$.

A) $\dfrac{4-x}{3}$ B) $\dfrac{4-3x}{3}$ C) $\dfrac{3x-4}{3}$ D) $\dfrac{3x+4}{3}$ E) $\dfrac{x+4}{3}$

8. If $f(x) = 2x - 8$, find $f^{-1}(2x)$.

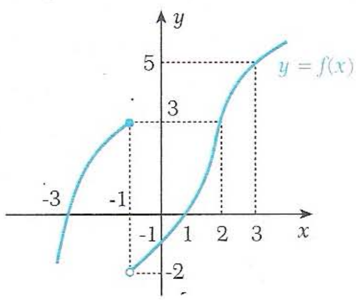
A) $4x + 16$ B) $4x + 8$ C) $\dfrac{x+6}{2}$ D) $x - 4$ E) $x + 4$

9. If $f(x) = x + 2$ and $g(x) = 3x + 1$, find $(g^{-1} \circ f^{-1})(x)$.

A) $3x + 3$ B) $3x + 7$ C) $\frac{x}{3} - 1$

D) $\frac{x}{3} + 2$ E) $\frac{x-2}{3}$

10. Using the figure find $(f \circ f \circ f)(0)$.



A) 5 B) 3 C) 2 D) 1 E) –1

11. If $f(x) = x^2$, $g(x) = x + 1$ and $h(x) = 3x - 2$, find $(h \circ g \circ f)(x)$

A) $3x^2 + 1$ B) $3x^2 - 1$ C) $3x^2 + 3$

D) $9x^2 - 6x + 1$ E) $9x^2 + 6x + 1$

12. If $f(x) = \dfrac{3x-2}{2x+1}$ and $g(x+1) = \dfrac{2x-1}{x+1}$, find $(g \circ f)(3)$.

A) –3 B) –1 C) 0 D) 1 E) 3

13. If $g(x) = 2x + 3$ and $(f \circ g)(x) = 6x + 8$, find $f(x)$.

A) $3x - 1$ B) $12x + 11$ C) $2x + 8$

D) $6x - 3$ E) $3x - 4$

14. Given the graph of $y = x^3$ which one of the following is correct for the graph of $y = \frac{1}{2}(3x-6)^3$?

A) It is shrinked in the y direction by a factor of ½.

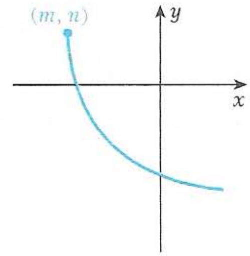
B) It is stretched in the y direction by a factor of 2.

C) It is stretched in the x direction by a factor of 3.

D) It is shifted in 6 units to the left.

E) It is shifted 6 units to the right.

15. Which one of the equations below is correct equation for the graph on the right?



A) $y = a\sqrt{x-m} + n,\ a > 0$

B) $y = a\sqrt{m-x} + n,\ a > 0$

C) $y = a\sqrt{x-m} + n,\ a < 0$

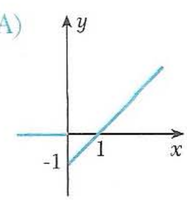
D) $y = a\sqrt{m-x} + n,\ a < 0$

E) $y = a\sqrt{m-x} + n,\ a < 0$

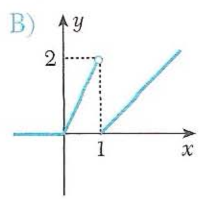
16. If $f(x) = \begin{cases} -1, & x < 0 \\ x-1, & x \geq 0 \end{cases}$ and $g(x) = \begin{cases} 1, & x < 0 \\ x+1, & 0 \leq x < 1, \\ 0, & x \geq 1 \end{cases}$

which one is the graph of $(f + g)(x)$?

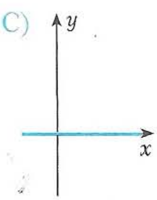
A)



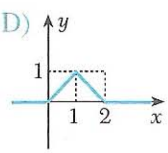
B)



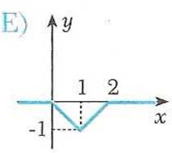
C)



D)



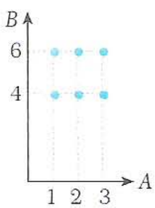
E)

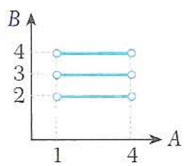


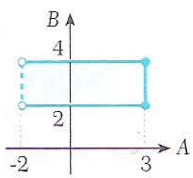
ANSWERS TO EXERCISES

EXERCISES 1

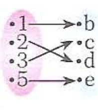
1. a. $x = -1$, $y = 4$ b. $x = 1$, $y = 0$ 2. a. $\{3\}$ b. 1 c. $\{1, 3, 5, 7, 8, 9, 10\}$ d. $\{1, 2, 3, 4, 5, 7, 8, 9, 10\}$ e. $\{3\}$ f. 2 3. a. $(-1, 1]$, $(-3, 2)$ b. $(2, 5)$, $(-3, \infty)$ 4. a. $A = \{1, 2\}$, $B = \{a, b, c\}$ b. $A = \{1, 2\}$, $B = [2, 4]$ 5. a. $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$, $\{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$, $\{(1, -1), (1, 1), (2, -1), (2, 1), (3, -1), (3, 1)\}$, $\{(-1, 1), (-1, 2), (-1, 3), (1, 1), (1, 2), (1, 3)\}$ b. {(December, December), (December, January), (December, February), (January, December), (January, January), (January, February), (February, December), (February, January), (February, February)}, $\{(2, 2), (2, 5), (2, 6), (5, 2), (5, 5), (5, 6), (6, 2), (6, 5), (6, 6)\}$, {(December, 2), (December, 5), (December, 6), (January, 2), (January, 5), (January, 6), (February, 2), (February, 5), (February, 6)}, {(2, December), (2, January), (2, February), (5, December), (5, January), (5, February), (6, December), (6, January), (6, February)}

6. a.


b.


c.


7. a. $\{(0, 0), (1, 1), (2, 4), (3, 9)\}$ b. $\{(-2, -5/3), (-1, -1), (0, -1/3), (1, 1/3), (2, 1), (3, 5/3)\}$

8. a.


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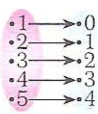

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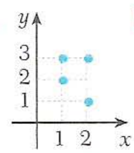

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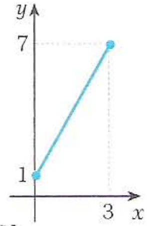

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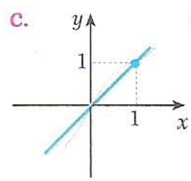

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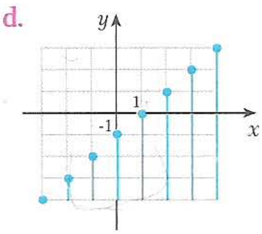
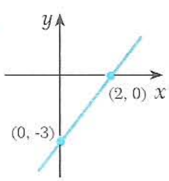


10. a. {(December, winter), (September, fall), (June, summer), (March, spring)}, {December, September, June, March}, {winter, fall, summer, spring} b. $x = 2y - 3$, $[-43, 65]$, $[-20, 34]$ c. $x = y^2 + 2$, $[2, 11]$, $[-3, 3]$

11. a.


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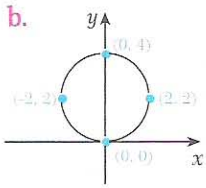

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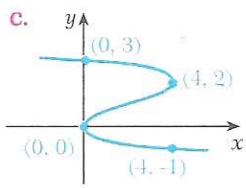

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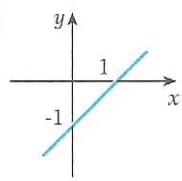
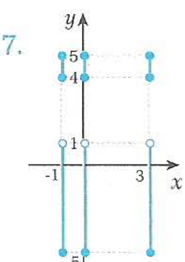


13. $\{(a, a), (a, b), (b, b), (b, c), (c, a), (c, b), (c, c), (c, d), (d, a), (d, d)\}$ 14. 84

15. use the fact that $n(A) + n(B)$ includes $n(A \cap B)$ twice. 16. $\{(1, a, \text{blue}), (1, a, \text{red}), (1, b, \text{blue}), (1, b, \text{red}), (2, a, \text{blue}), (2, a, \text{red}), (2, b, \text{blue}), (2, b, \text{red}), (3, a, \text{blue}), (3, a, \text{red}), (3, b, \text{blue}), (3, b, \text{red})\}$, use three coordinate axes that are perpendicular to each other in the space

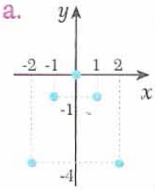
17.


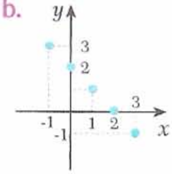
EXERCISES 2

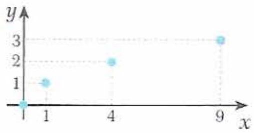
1. a. function b. not c. not 2. a. $f(x) = 5x + 2$ b. $f(x) = x^2 - 2x$ c. $f(x) = \frac{x}{2} + 3x^3$

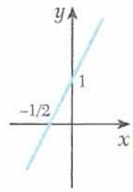
3. a. multiply by 2, then subtract 4 b. add 1, then take the square root of all c. divide by three times itself increased by 1

4. a. -3, 33, $2x^2 + 5x$, $-2x^2 + 5x$, $2x - 5\sqrt{x}$, $2x^2 + 4hx - 5x + 2h^2 - 5h$ b. $-\frac{1}{2}$, $\frac{11}{18}$, $\frac{9x-1}{9x^2+2}$, $-\frac{6x+2}{x^2+2}$, $\frac{3x^2-1}{x^4+2}$, $\frac{6a+3b-1}{4a^2+4ab+b^2+2}$ c. $-\frac{2}{3}$, undefined, $\frac{4u^2+1}{8u^3+4u}$, $\frac{x^2+2x+2}{x^3+3x^2+5x+3}$, $\frac{u^4+1}{u^6+2u^2}$, $\frac{u^4+2u^2+1}{u^6+4u^4+4u^2}$ 5. $-\frac{8}{3}$

6. a.


b.


c.


d.


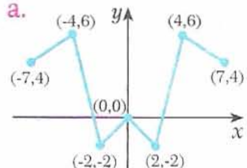
7. a. $\frac{7}{3}$; -7 b. 6, -1; -6 c. 2; $-\frac{4}{5}$ d. 21; -3

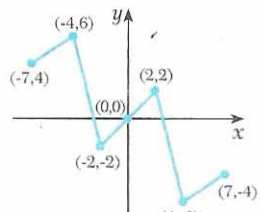
8. a. function b. not c. function d. function e. not f. not

9. a. $\mathbb{R}$ b. $\mathbb{R}$ c. $(-\infty, 4) \cup (4, \infty)$ d. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ e. $[2, \infty)$ f. $(-\infty, -3) \cup (-3, 0) \cup (0, \infty)$ g. $[5, 6) \cup (6, \infty)$ h. $[5, 10]$ i. $\left(\frac{1}{5}, 1\right]$ j. $[-1, 1) \cup \left(1, \frac{3}{2}\right]$ k. $\left[3, \frac{10}{3}\right) \cup \left(\frac{10}{3}, \infty\right)$ 10. a. $\left[-\frac{3}{2}, \frac{5}{4}\right]$ b. $[-4, -1) \cup \left(-1, \frac{5}{3}\right)$ c. $(-1, 1) \cup (1, \infty)$ d. $(-\infty, -2) \cup (4, \infty)$ e. $[-1, 2]$ f. $[1, 2] \cup \{0\}$ g. $(0, 1) \cup (1, 3)$ h. $[-5, -1) \cup (1, 5]$ i. $(-\infty, -1] \cup [11, \infty)$ j. $[-5, -2] \cup [2, 3) \cup (3, 5]$ 11. a. $f(x) = \sqrt{\frac{3-x}{x-5}}$ b. $f(x) = \frac{\sqrt{-x^2-4x}}{x^2+2x}$ c. $f(x) = \sqrt{(x-3)^2(x+2)(1-x)}$

12. a. $-\frac{4}{3}$ b. -4, 4 c. $\varnothing$ 13. a. odd b. neither c. even d. neither e. even f. even g. odd h. odd i. neither j. odd

14. a. even b. odd c. even d. odd e. neither f. even

15. a.




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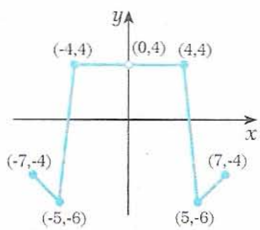
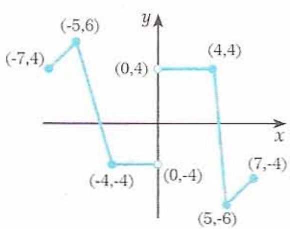
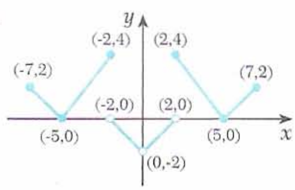
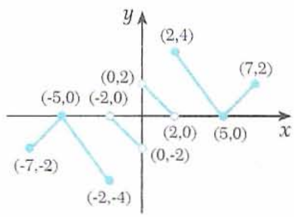
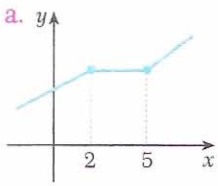
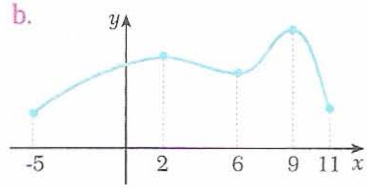


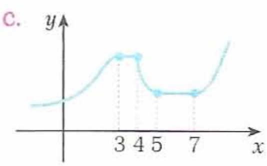


c.




16. a.


b.


c.


17. consider $f(x_2) - f(x_1)$ where $x_1 < x_2$ on the given interval 18. 3 19. -2

20. a. $\mathbb{R}$, $\mathbb{R}$ b. increasing on $(-\infty, -2]$ and $[5, \infty)$, decreasing on $[-2, 5]$ c. -6, 3, 6; 3 d. -4, 5, 8 e. $(-\infty, -6) \cup (3, 6)$

21. a. $(-\infty, -10) \cup (-10, 10) \cup (10, \infty)$, $[-7, \infty)$ b. $-5, 5$ c. 2 d. undefined e. $[-5, 0) \cup (0, 5]$ f. $(-\infty, -10) \cup (10, \infty)$ g. $(-10, -6] \cup \{0\} \cup [6, 10)$ h. even i. increasing on $[3, 6]$ and $(10, \infty)$, decreasing on $(-\infty, -10)$ and $[-6, -3]$, constant on $(-10, -6]$, $[-3, 0)$, $(0, 3]$ and $[6, 10)$ j. $-7, \infty$ k. $[-3, 0) \cup (0, 3]$ l. $(-6, -3) \cup (3, 6)$ 22. 0 23. $d(t) = 15t$ 24. a. $F(x) = 1.8x + 32$ b. 1.8

25. a. $V(t) = 10t$ b. $V(h) = \begin{cases} 225h^2 & \text{if } 0 \le h \le 2 \\ 1200h - 1500 & \text{if } 2 < h \le 3 \end{cases}$ c. $h(t) = \begin{cases} \dfrac{\sqrt{10t}}{15} & \text{if } 0 \le t \le 90 \\ 2 + \dfrac{t-90}{120} & \text{if } 90 < t \le 210 \end{cases}$

26. until his twenties the person gained weight and kept his weight constant until his forties, then he lost weight because of a diet or an illness and after his mid-forties he started to gain weight again. 27. he left home at 8 a.m. and during the day visited four places, probably for a sale, and at 6 p.m. left for home. 28. a. $(-\infty, 4]$ b. $[3, \infty)$ c. $\left(0, \dfrac{1}{6}\right]$ d. $(-\infty, 1) \cup (1, \infty)$ 29. 100 30. 0, 2, -1 31. -8 32. decreasing on $\left(-\infty, -\dfrac{1}{5}\right]$, increasing on $\left[-\dfrac{1}{5}, \infty\right)$

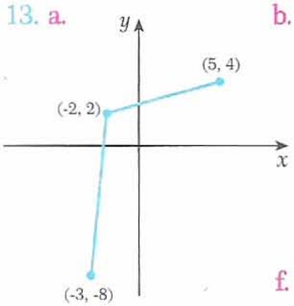
33. $-x^3 - 3x$ 34. $\dfrac{13}{9}$ 35. 12 36. $\dfrac{2-x^2}{3x}$

EXERCISES 3

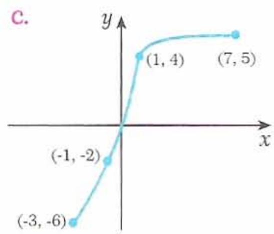
1. a. $x^2 + x$, $-x^2 + x + 2$, $x^3 + x^2 - x - 1$, $\dfrac{1}{x-1}$ with domain $\mathbb{R} \setminus \{-1, 1\}$ b. $x^3 + 4x^2 + 5x$, $x^3 + 2x^2 - 5x$, $x^5 + 8x^4 + 15x^3$, $\dfrac{x^2+3x}{x+5}$ with domain $\mathbb{R} \setminus \{-5, 0\}$ c. $x + 3 + \sqrt{x+2}$, $x + 3 - \sqrt{x+2}$, $(x+3)\sqrt{x+2}$, $\dfrac{x+3}{\sqrt{x+2}}$ d. $\sqrt{x-1} + \sqrt{x+1}$, $\sqrt{x-1} - \sqrt{x+1}$, $\sqrt{x-1} \cdot \sqrt{x+1}$, $\dfrac{\sqrt{x-1}}{\sqrt{x+1}}$ 2. a. 25 b. undefined c. -11 3. a. x, $\dfrac{x^2+x+1}{x^2+x-1}$, $\dfrac{2x^2+2x}{x^2-2x+1}$, $x^4 + 2x^3 + 2x^2 + x$ b. $x^4 - 4x^3 + 4x^2$, x^2, $x^2 - 2x + 2$, $x + 2$ c. $\dfrac{7x+6}{2x+3}$, $\dfrac{11x}{x+3}$, $\dfrac{5x+3}{4x+9}$, $\dfrac{10x-3}{13-3x}$

4. a. $f(x) = \sqrt{x}$, $g(x) = 2x - 4$ b. $f(x) = x^7$, $g(x) = \dfrac{x-4}{5}$ c. $f(x) = x^3 - x + 4$, $g(x) = 2x^2 - x$ 5. a. $\dfrac{41}{11}$ b. -1 c. 4 d. 9 6. 2 7. 2 8. x 9. a. not b. not c. one-to-one 10. a. one-to-one b. one-to-one c. not d. one-to-one e. one-to-one 11. a. one-to-one b. one-to-one c. not 12. a. $\dfrac{6-x}{4}$ b. $\dfrac{x+4}{x-3}$ c. $\sqrt[3]{\dfrac{x+5}{4}}$ d. $\left(\dfrac{x-9}{3}\right)^5 + 1$

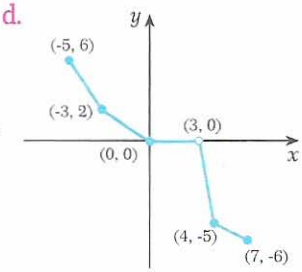
13. a.



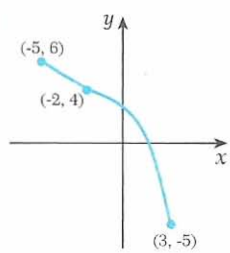
b. inverse function does not exist c.



d.



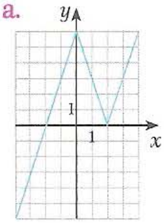
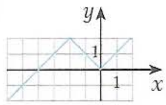
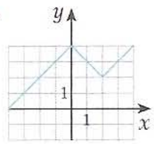
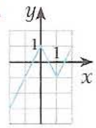
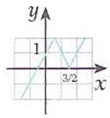
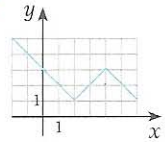
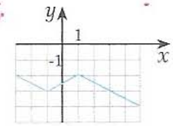
e.

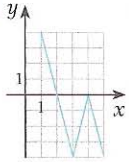
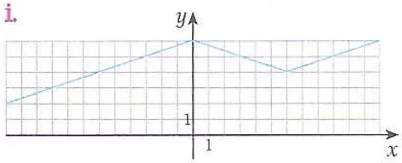
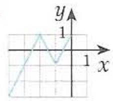
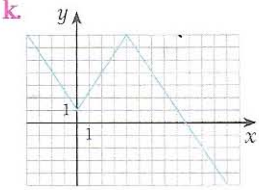


f. inverse function does not exist

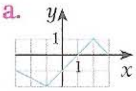
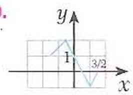
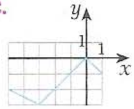
14. a. 5 b. 0 c. 3 d. –14 e. 7 15. a. $\frac{2x-1}{x-2}$ b. $\frac{22x-1}{3-x}$ c. $\frac{6x-5}{2x+1}$ 16. a. $\frac{3x-7}{2}$ b. $\frac{2x-1}{2-x}$ c. x d. $\frac{2x}{3x-2}$ e. $2x-11$ f. $\frac{3x-7}{x-5}$ g. $\frac{x-1}{2x-4}$ 17. a. 1 b. –1 c. 1 d. $-\frac{17}{16}$ 18. a. 1 b. –2 19. a. find two different x-values that give the same y-value b. assume that for $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$ 20. –2 21. Compare the functions when the argument is x and $-x$ 22. Compare the functions when the argument is x and $-x$ 23. a. $-\sqrt{x+9}-3$ b. $\sqrt{x+11}+4$ 24. Try to compose each of the sides with $g \circ f$

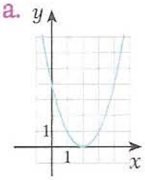
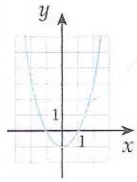
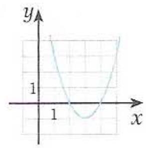
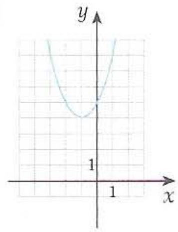
EXERCISES 4

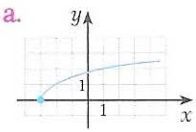
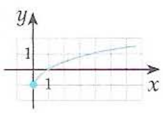
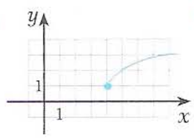
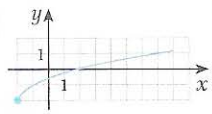
1. a.  b.  c.  d.  e.  f.  g. 

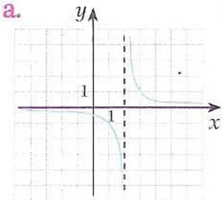
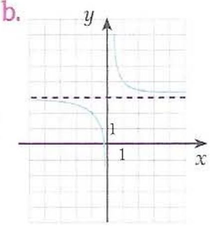
h.  i.  j.  k. 

2. 1: $-f(x-1)-2$,
2: $-f(x+2)-2$,
3: $f(x+2)$,
4: $-f(x+1)+3$,
5: $-2f(x+5)+2$,
6: $\frac{1}{2}f(x+3)-\frac{1}{2}$

3. a.  b.  c. 

4. a.  b.  c.  d. 

5. a.  b.  c.  d. 

6. a.  b. 

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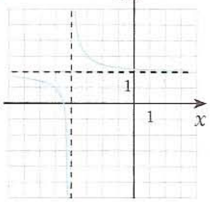

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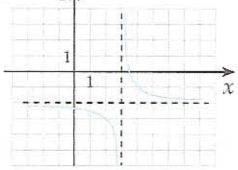

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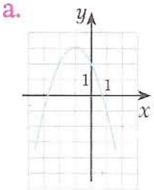

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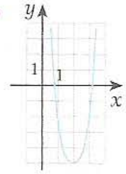

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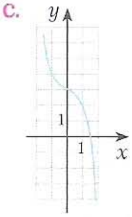

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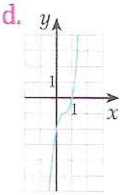

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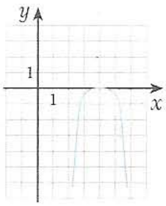

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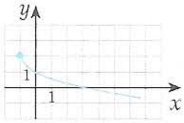

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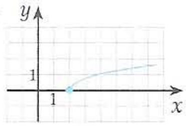

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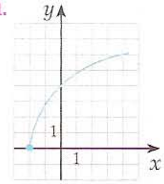

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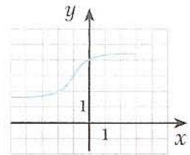

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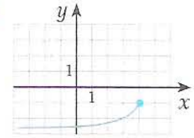

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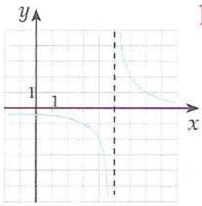

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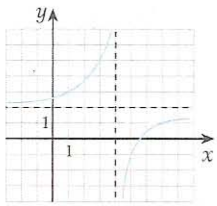

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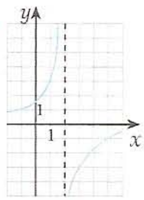

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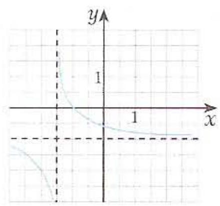

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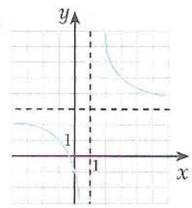

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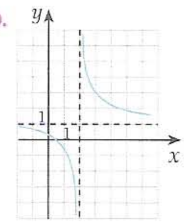

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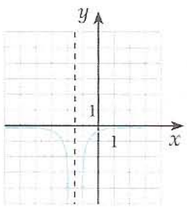

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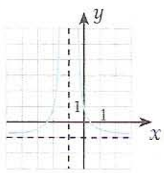

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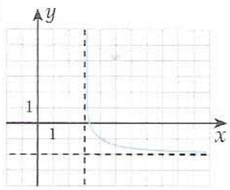

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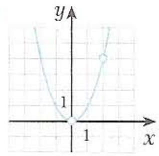

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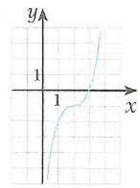

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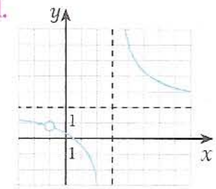

9. a. $y=\dfrac{-3}{x}-1$ b. $y=-3(x+1)^3+1$ c. $y=\dfrac{-5}{(x-1)^2}-2$ d. $y=\sqrt{-x+1}+2$ e. $y=-2\sqrt{x-1}+2$ f. $y=\dfrac{1}{3}(x+3)^2-4$

g. $y=\dfrac{1}{x^2}-1$ h. $y=\dfrac{2}{x-4}+2$ i. $y=\dfrac{5}{8}(x-2)^3+5$ j. $y=\dfrac{1}{2}(x-3)^2-2$

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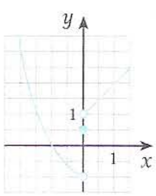

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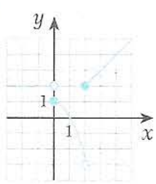

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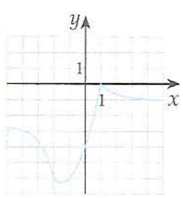

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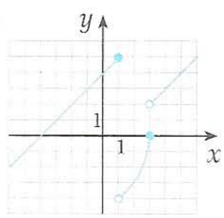

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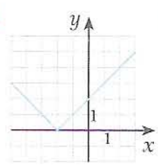

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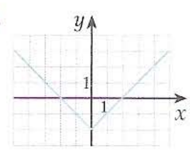

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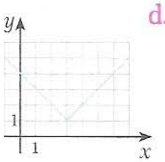

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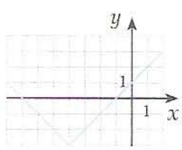

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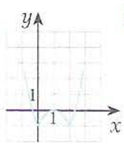

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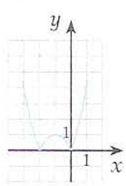

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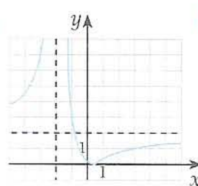

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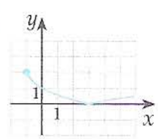

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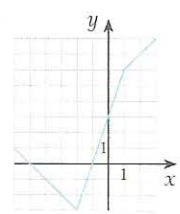
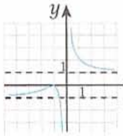
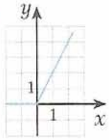
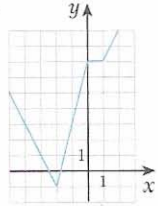
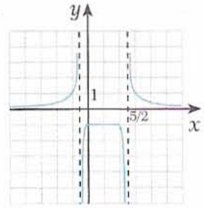
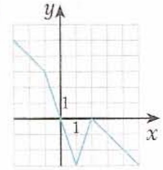
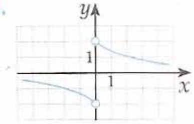
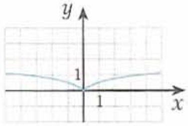
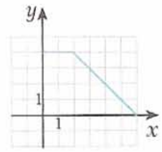
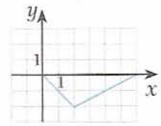
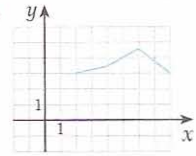
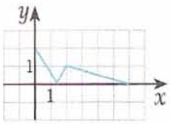
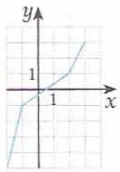
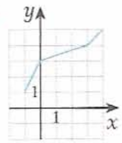
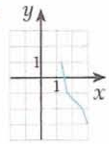
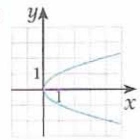
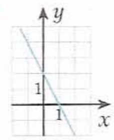
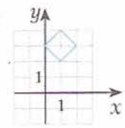


f.  g.  13. a.  b.  c.  14. 

15.  16. $(-\infty, -3) \cup (1, 2)$ 17. 3 18. one 19. a.  b.  c. 

d.  20. a.  b.  c.  21. a.  b.  c. 

ANSWERS TO TESTS

TEST 1

1.	B	9.	C
2.	E	10.	C
3.	A	11.	A
4.	B	12.	C
5.	D	13.	E
6.	B	14.	B
7.	D	15.	E
8.	A	16.	C

TEST 2

1.	C	9.	C
2.	E	10.	C
3.	B	11.	B
4.	C	12.	C
5.	C	13.	C
6.	C	14.	B
7.	E	15.	C
8.	D	16.	B

TEST 3

1.	A	9.	C
2.	E	10.	A
3.	C	11.	A
4.	E	12.	B
5.	D	13.	A
6.	B	14.	A
7.	B	15.	C
8.	E	16.	B

GLOSSARY

A

abscissa: x-coordinate of a point.

absolute value: non-negative difference of a number from zero.

analytic plane: (also known as xy-plane or coordinate plane) the plane on which a coordinate system consisting of one horizontal axis (x-axis) and one vertical axis (y-axis) is constructed.

argument: (also known as independent variable) a variable in a mathematical expression whose value determines the dependent variable; if $f(x) = y$, x is the independent variable.

asymptote: a virtual line which the graph of a function approaches nearer and nearer, but never touches.

axes: plural of axis.

C

cartesian: of or regarding to Descartes.

cartesian product: set of all ordered pairs whose first component is from a set and whose second component is from another set.

component: each of elements a and b of an ordered pair (a, b).

composition: an operation which combines functions $f(x)$ and $g(x)$ as $f(g(x))$.

constant function: a function whose value is a constant number on its domain.

continuous function: a function whose graph contains no breaks or gaps.

coordinate plane: see *analytic plane*.

coordinates (of a point): corresponding values of a point on the x and y-axes.

cubic function: a polynomial function whose highest degree is three.

D

decreasing function: a function whose value decreases on its domain.

Descartes: René Descartes (1596-1650), French mathematician and philosopher.

discontinuous function: a function whose graph contains breaks or gaps.

domain (of a function): the set of values of the independent variable for which a function is defined.

E

element: each object that form a set.

even function: a function for which $f(-x) = f(x)$ for any value on its domain.

F

formulae: plural of formula.

function: a relation which assigns to each element in its domain just one element from its range.

H

horizontal line test: a test to understand whether a function is one-to-one or not (a function is one-to-one if and only if no horizontal line crosses its graph at more than one point).

horizontal shift: moving a graph to the left or right without changing its shape.

horizontal shrink: changing the shape of a graph to a smaller scale horizontally.

horizontal stretch: changing the shape of a graph to a bigger scale horizontally.

hyperbola: the graph of the function of the form $y = 1/x$.

I

increasing function: a function whose value increases on its domain.

independent variable: see *argument*.

integer: any number which is a member of the set $\{...-3, -2, -1, 0, 1, 2, 3, ...\}$.

intercept: the point at which graph intersects an axis.

intersection: the set of elements common to two or more sets.

interval: a set containing all real numbers or points between two given numbers or points, called the endpoints. A closed interval includes the endpoints, but an open interval does not.

inverse function: the function obtained by expressing the dependent variable of one function as the independent variable of another.

irrational number: any real number which cannot be expressed as a fraction.

N

natural number: any number which is a member of the set $\{1, 2, 3, ...\}$.

O

odd function: a function for which $f(-x) = -f(x)$ for any value on its domain.

one-to-one function: a function for which $f(x_1) \neq f(x_2)$ for any $x_1 \neq x_2$.

onto function: a function for which, for any y in its range, there is at least one x from the domain such that $f(x) = y$.

ordered pair: an element of the form (a, b) where a is the first component and b is the second component.

ordinate: y-coordinate of a point.

origin: the point of intersection of the coordinate axes.

P

parabola: the graph of a quadratic function

piecewise defined function: (also known as hybrid function) a function which is defined by different formulae in different parts of its domain.

plot: to locate on an anaytic plane.

polynomial function: a function which is written as a sum of terms.

Q

quadrant: any of the four areas into which analytic plane is divided by axes.

R

range (of a function): the set of all possible values which a function can take.

rational function: a function which is written as the quotient of polynomials.

rational number: any real number which can be expressed as a fraction.

real number: any rational or irrational number.

reflection: changing shape of a graph as reflected by a mirror.

relation: a set of ordered pairs.

root (of a function): (also known as zero of a function) all x-values for which a function is equal to zero.

root function: a function consisting of square, cubic or higher degree root expression.

S

set: a collection of objects.

slope: inclination of a line with respect to positive x-axis; m is the slope of the line $y = mx + c$.

square root function: a function consisting of square root expression.

symmetric (with respect to origin): exact correspondence of each point of a graph on the other end of the extension line through origin.

symmetric (with respect to the x-axis): exact reflection of a graph on opposite sides of the x-axis.

symmetric (with respect to the y-axis): exact reflection of a graph on opposite sides of the y-axis.

T

transformation (of graph): changing the shape and location of a graph.

translation (of graph): moving a graph on an analytic plane without changing its shape.

truncus: the graph of the function of the form $y = 1/x^2$.

U

union: a set containing all and only the members of two or more given sets.

V

vertical line test: a test to understand whether a relation is a function or not (a set of points in the coordinate plane is the graph of a function if and only if no vertical line crosses the graph at more than one point).

vertical shift: moving a graph upwards or downwards without changing its shape.

vertical shrink: changing shape of a graph to a smaller scale vertically.

vertical stretch: changing shape of a graph to a bigger scale vertically.

X

xy-plane: see *analytic plane*.

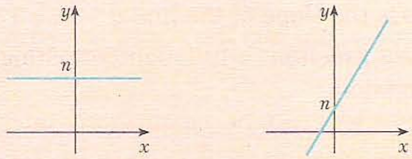
Z

zero (of a function): see *root (of a function)*.

BASIC GRAPHS

Linear functions

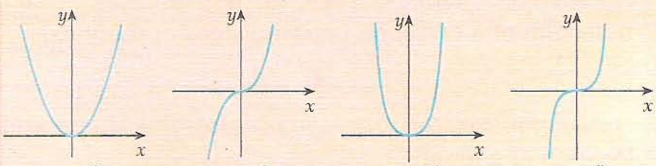
$$f(x) = mx + n$$



$y = n$ $\qquad$ $y = mx + n$

Power functions

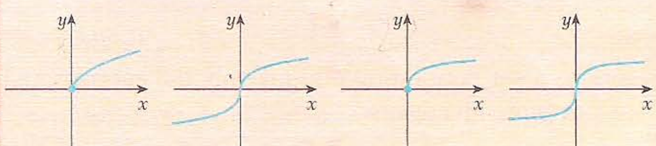
$$f(x) = x^n$$



$y = x^2$ $\qquad$ $y = x^3$ $\qquad$ $y = x^4$ $\qquad$ $y = x^5$

Root functions

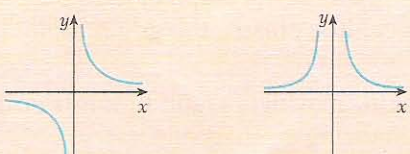
$$f(x) = \sqrt[n]{x}$$



$y = \sqrt{x}$ $\qquad$ $y = \sqrt[3]{x}$ $\qquad$ $y = \sqrt[4]{x}$ $\qquad$ $y = \sqrt[5]{x}$

Reciprocal functions

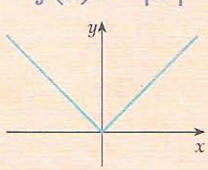
$$f(x) = 1/x^n$$



$y = \frac{1}{x}$ $\qquad$ $y = \frac{1}{x^2}$

Absolute value function

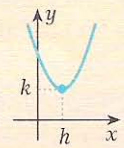
$$f(x) = |x|$$



$y = |x|$

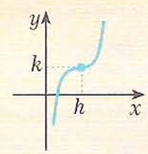
Parabola

$$y = a(x-h)^2 + k$$

Graph	Domain	Range	Property
	$\mathbb{R}$	if $a > 0$, $[k, \infty)$ if $a < 0$, $(-\infty, k]$	vertex point at (h, k)

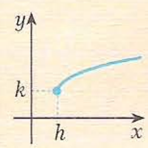
Cubic

$$y = a(x-h)^3 + k$$

Graph	Domain	Range	Property
	$\mathbb{R}$	$\mathbb{R}$	stationary point of inflection at (h, k)

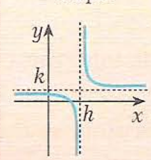
Square Root

$$y = a\sqrt{x-h} + k$$

Graph	Domain	Range	Property
	$[h, \infty)$	if $a > 0$, $[k, \infty)$ if $a < 0$, $(-\infty, k]$	end point at (h, k)

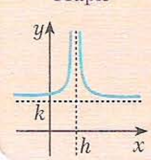
Hyperbola

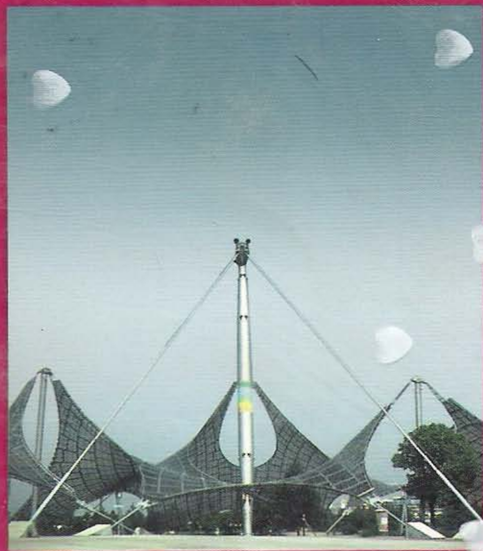
$$y = \frac{a}{x-h} + k$$

Graph	Domain	Range	Property
	$\mathbb{R} \setminus \{h\}$	$\mathbb{R} \setminus \{k\}$	asymptotes $y = k$, $x = h$

Truncus

$$y = \frac{a}{(x-h)^2} + k$$

Graph	Domain	Range	Property
	$\mathbb{R} \setminus \{h\}$	(k, ∞)	asymptotes $y = k$, $x = h$



FUNCTIONS

MODULAR SYSTEM

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